



# The $p$ -rank strata of the moduli space of hyperelliptic curves<sup>☆</sup>

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## Abstract

We prove results about the intersection of the  $p$ -rank strata and the boundary of the moduli space of hyperelliptic curves in characteristic  $p \geq 3$ . This yields a strong technique that allows us to analyze the stratum  $\mathcal{H}_g^f$  of hyperelliptic curves of genus  $g$  and  $p$ -rank  $f$ . Using this, we prove that the endomorphism ring of the Jacobian of a generic hyperelliptic curve of genus  $g$  and  $p$ -rank  $f$  is isomorphic to  $\mathbb{Z}$  if  $g \geq 4$ . Furthermore, we prove that the  $\mathbb{Z}/\ell$ -monodromy of every irreducible component of  $\mathcal{H}_g^f$  is the symplectic group  $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$  if  $g \geq 3$ , and  $\ell \neq p$  is an odd prime (with mild hypotheses on  $\ell$  when  $f = 0$ ). These results yield numerous applications about the generic behavior of hyperelliptic curves of given genus and  $p$ -rank over finite fields, including applications about Newton polygons, absolutely simple Jacobians, class groups and zeta functions.

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## 1. Introduction

Suppose  $C$  is a smooth connected projective hyperelliptic curve of genus  $g \geq 1$  over an algebraically closed field  $k$  of characteristic  $p \geq 3$ . The Jacobian  $\mathrm{Pic}^0(C)$  is a principally polarized

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abelian variety of dimension  $g$ . The number of physical  $p$ -torsion points of  $\text{Pic}^0(C)$  is  $p^f$  for some integer  $f$ , called the  $p$ -rank of  $C$ , with  $0 \leq f \leq g$ .

Let  $\mathcal{H}_g$  be the moduli space over  $k$  of smooth connected projective hyperelliptic curves of genus  $g$ ; it is a smooth Deligne–Mumford stack over  $k$ . The  $p$ -rank induces a stratification  $\mathcal{H}_g = \bigcup \mathcal{H}_g^f$  by locally closed reduced substacks  $\mathcal{H}_g^f$ , whose geometric points correspond to hyperelliptic curves of genus  $g$  and  $p$ -rank  $f$ .

In this paper, we prove three cumulative results about  $\mathcal{H}_g^f$ . The first is about the boundary of  $\mathcal{H}_g^f$ ; specifically, when  $g \geq 2$ , we prove that the boundary of every irreducible component of  $\mathcal{H}_g^f$  contains the moduli point of some singular curve which is a tree of elliptic curves and which has  $p$ -rank  $f$ . The second is that the Jacobian of a generic geometric point of  $\mathcal{H}_g^f$  has endomorphism ring  $\mathbb{Z}$  if  $g \geq 4$ . The third is that, for an odd prime number  $\ell$  distinct from  $p$ , the  $\ell$ -adic monodromy group of every irreducible component of  $\mathcal{H}_g^f$  is the symplectic group  $\text{Sp}_{2g}(\mathbb{Z}/\ell)$  (with mild hypotheses on  $\ell$  when  $f = 0$ ). Heuristically, this means that  $p$ -rank constraints alone do not force the existence of extra automorphisms (or other algebraic cycles) on a family of hyperelliptic curves.

We now state the results of this paper more precisely.

**Theorem 3.11(c).** *Suppose  $p$  is an odd prime,  $g \geq 2$ , and  $0 \leq f \leq g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ , the  $p$ -rank  $f$  stratum in  $\mathcal{H}_g$ . Then the closure  $\bar{S}$  of  $S$  in  $\overline{\mathcal{H}}_g$  contains the moduli point of some tree of  $g$  elliptic curves, of which  $f$  are ordinary and  $g - f$  are supersingular.*

Our proof does not yield much information on the structure of the tree in Theorem 3.11; however, once the tree's structure is fixed, we prove that any choice of labeling of  $f$  components as ordinary and  $g - f$  components as supersingular will occur for some moduli point in  $\bar{S}$ .

Theorem 3.11 yields a powerful technique to analyze  $\mathcal{H}_g^f$ . Using it, we prove the following two results.

**Theorem 4.6.** *Suppose  $p$  is an odd prime,  $g \geq 4$ , and  $0 \leq f \leq g$ . Suppose  $\xi$  is a geometric generic point of  $\mathcal{H}_g^f$  with Jacobian  $\mathcal{X}_{g,\xi} = \text{Pic}^0(C_{g,\xi})$ . Then  $\text{End}(\mathcal{X}_{g,\xi}) \cong \mathbb{Z}$  and thus  $\mathcal{X}_{g,\xi}$  is simple.*

The third theorem requires some notation. Let  $S$  be a connected stack over  $k$ , and let  $s$  be a geometric point of  $S$ . Let  $C \rightarrow S$  be a relative smooth proper curve of genus  $g$  over  $S$ . Then  $\text{Pic}^0(C)[\ell]$  is an étale cover of  $S$  with geometric fibers isomorphic to  $(\mathbb{Z}/\ell)^{2g}$ . The fundamental group  $\pi_1(S, s)$  acts linearly on the fiber  $\text{Pic}^0(C)[\ell]_s$ , and the monodromy group  $M_\ell(C \rightarrow S, s)$  is the image of  $\pi_1(S, s)$  in  $\text{Aut}(\text{Pic}^0(C)[\ell]_s)$ . For the third main result, we determine  $M_\ell(S) := M_\ell(C \rightarrow S, s)$ , where  $S$  is an irreducible component of  $\mathcal{H}_g^f$  and  $C \rightarrow S$  is the universal curve.

**Theorem 5.2–5.7.** *Suppose  $p$  is an odd prime,  $g \geq 1$ , and  $0 \leq f \leq g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ .*

- (i) *If  $1 \leq f \leq g$  and  $\ell$  is an odd prime distinct from  $p$ , then  $M_\ell(S) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$ .*
- (ii) *If  $f = 0$  and if  $g \geq 4$ , then  $M_\ell(S) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$  for all primes  $\ell$  outside a finite set which depends only on  $p$ .*

We also prove that the  $\ell$ -adic monodromy group is  $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$  in the situation of Theorem 5.2–5.7. (Note that the case of ordinary hyperelliptic curves, i.e., when  $f = g$ , follows directly from previous work, see J.K. Yu [unpublished], [3, Thm. 3.4], or [19, Thm. 5.1].) In addition, we determine the  $p$ -adic monodromy group of components of  $\mathcal{H}_g^f$  when  $f \geq 1$  (Proposition 5.4).

This paper is a natural generalization of our paper [4], which is about  $\mathcal{M}_g^f$ , the  $p$ -rank strata of the moduli space of curves. The two papers share essential similarities, but there are several new phenomena for hyperelliptic  $p$ -rank strata which increase the difficulty of the proofs and influence the final results in this paper. First, the boundary component  $\Delta_0$  is more complicated for  $\overline{\mathcal{H}}_g$  than for  $\overline{\mathcal{M}}_g$ . Second, for a singular hyperelliptic curve which is formed as a chain of two hyperelliptic curves of smaller genera, the set of possibilities for the location of the ordinary double point is discrete. These two facts play a key role in the degeneration arguments found in the proof of Theorem 3.11. The third issue, which arises in a base case when  $f = 0$ , is that the stratum  $\mathcal{H}_3^0$  is not nearly as well understood as  $\mathcal{M}_3^0$ .

The second and third main results of the paper rely on Theorem 3.11 because the proofs use degeneration in order to proceed by induction on the genus. For the inductive step, Theorem 3.11 implies that the closure of every component  $S$  of  $\mathcal{H}_g^f$  in  $\overline{\mathcal{H}}_g$  intersects the stratum  $\Delta_{1,1}$  of the boundary of  $\mathcal{H}_g$  (Corollary 3.14). This is used in the proof of Theorem 4.6 to show that the endomorphism ring of  $\mathrm{Pic}^0(\mathcal{C}_{g,S})$  acts diagonally on the Tate module. It is also used in the proof of Theorem 5.2–5.7 to show that the monodromy group of  $S$  contains two non-identical copies of  $\mathrm{Sp}_{2g-2}(\mathbb{Z}/\ell)$ .

There are two base cases needed in this paper. The first, when  $g = 2$  and  $f \geq 1$ , uses facts about  $\mathcal{H}_2^f$  from a special case of [8, Prop. 4.4]. The second, when  $g = 3$  and  $f = 0$ , we found somewhat intractable. For this reason, the results required a novel analysis of endomorphism algebras of generic Jacobians of small genus with  $p$ -rank zero. Applying Theorem 3.11 and [31, Thm. 1.12], we were able to constrain the possibilities (Lemma 4.1) for the endomorphism algebra of the Jacobian of a generic hyperelliptic curve of genus 3 with  $p$ -rank 0. This, combined with an understanding of abelian varieties of Mumford type, allowed us enough leverage to understand the case when  $g = 4$  and  $f = 0$ . In particular, it allowed us to determine the endomorphism ring of a generic hyperelliptic curve of genus 4 and  $p$ -rank 0 and to compute the monodromy group of components of  $\mathcal{H}_4^0$ .

This paper also contains multiple applications about hyperelliptic curves of given genus and  $p$ -rank. For example, results on Newton polygons of (hyperelliptic) curves for arbitrary  $g$  and  $p$  are notoriously elusive. As a consequence of Theorem 3.11, we give an application about Newton polygons of hyperelliptic curves. The application relies on and generalizes [31, Thm. 1.12], which is the case  $g = 3$ .

**Corollary 3.16.** *Suppose  $p$  is an odd prime and  $g \geq 3$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^0$ , the  $p$ -rank 0 stratum in  $\mathcal{H}_g$ . Then  $S$  contains the moduli point of a curve whose Jacobian is not supersingular.*

We also give applications about class groups and zeta functions of hyperelliptic curves of given genus and  $p$ -rank over finite fields. These build upon [21, Thm. 9.7.13] and [23, Thm. 6.1].

**Applications:** Let  $\mathbb{F}$  be a finite field of characteristic  $p$ . Under the hypotheses of Theorem 5.2–5.7:

- (i) there is a hyperelliptic  $\mathbb{F}$ -curve  $C$  of genus  $g$  and  $p$ -rank  $f$  whose Jacobian is absolutely simple (Application 5.9);
- (ii) if  $|\mathbb{F}| \equiv 1 \pmod{\ell}$ , about  $\ell/(\ell^2 - 1)$  of the hyperelliptic  $\mathbb{F}$ -curves of genus  $g$  and  $p$ -rank  $f$  have a point of order  $\ell$  on their Jacobian (Application 5.11);
- (iii) for most hyperelliptic  $\mathbb{F}$ -curves  $C$  of genus  $g$  and  $p$ -rank  $f$ , the splitting field of the numerator of the zeta function of  $C$  achieves the maximal possible degree  $(2^g g!)$  over  $\mathbb{Q}$  (Application 5.13).

Here is an outline of the paper. Notation and background are found in Section 2. Section 3 contains the results about the boundary of the  $p$ -rank  $f$  stratum  $\mathcal{H}_g^f$  and the application to Newton polygons. This section ends with some open questions about the geometry of  $\mathcal{H}_g^f$ . For example, the number of irreducible components of  $\mathcal{H}_g^f$  is known only in special cases. The results about endomorphism rings are in Section 4 while the results about monodromy and the applications to absolutely simple Jacobians, class groups and zeta functions are in Section 5.

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## 2. Background

Let  $k$  be an algebraically closed field of characteristic  $p \geq 3$ . With the exception of Section 5.4, where we work over a finite field, all objects are defined on the category of  $k$ -schemes. Let  $\ell$  be an odd prime distinct from  $p$ . We fix an isomorphism  $\mu_\ell \cong \mathbb{Z}/\ell$ .

### 2.1. Moduli spaces

For a natural number  $g$  consider the following well-known categories, each of which is fibered in groupoids over the category of  $k$ -schemes in the étale topology:

- $\mathcal{A}_g$  principally polarized abelian schemes of dimension  $g$ ;
- $\mathcal{M}_g$  smooth connected proper curves of genus  $g$ ;
- $\overline{\mathcal{M}}_g$  stable curves of genus  $g$ .

Each of these is a smooth Deligne–Mumford stack, and  $\overline{\mathcal{M}}_g$  is proper [11, Thm. 5.2].

A relative smooth connected proper curve  $C \rightarrow S$  of genus  $g$  is hyperelliptic if there exists an involution  $\iota \in \text{Aut}_S(C)$  such that each fiber  $C_s/\langle \iota_s \rangle$  is a rational curve; let  $\mathcal{H}_g$  denote the moduli stack of all such. Since a smooth curve admits at most one hyperelliptic involution,  $\mathcal{H}_g$  may be equivalently defined [38, 1.4] as the corresponding substack of  $\mathcal{M}_g$ .

Let  $\overline{\mathcal{H}}_g$  be the closure of  $\mathcal{H}_g$  in  $\overline{\mathcal{M}}_g$ . Thus there are the following categories:

- $\mathcal{H}_g$  smooth connected proper hyperelliptic curves of genus  $g$ ;
- $\overline{\mathcal{H}}_g$  stable hyperelliptic curves of genus  $g$ ;
- $\tilde{\mathcal{H}}_g$  stable hyperelliptic curves of genus  $g$ , along with a labeling of the smooth ramification locus (see [3, Section 2.2]).

Both  $\overline{\mathcal{H}}_g$  and  $\tilde{\mathcal{H}}_g$  are smooth proper Deligne–Mumford stacks, by [14, Thm. 3.2] and [3, Lemma 2.2], respectively. The forgetful map  $\varpi_g : \tilde{\mathcal{H}}_g \rightarrow \overline{\mathcal{H}}_g$  is étale and Galois, with cover-

ing group  $\mathrm{Sym}(2g+2)$ . If  $S \subset \overline{\mathcal{H}}_g$ , let  $\bar{S}$  be the closure of  $S$  in  $\overline{\mathcal{H}}_g$ . Let  $\mathcal{C}_g$  be the universal curve over  $\overline{\mathcal{H}}_g$ , i.e., the pullback to  $\overline{\mathcal{H}}_g$  of the universal curve over  $\overline{\mathcal{M}}_g$  (cf. [11, Thm. 5.2]).

For a natural number  $r$ , let  $\overline{\mathcal{M}}_{g;r}$  be the Deligne–Mumford stack of stable curves of genus  $g$  with  $r$  labeled smooth points. Let  $\overline{\mathcal{H}}_{g;r} = \overline{\mathcal{H}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g;r}$  and let  $\tilde{\mathcal{H}}_{g;r} = \tilde{\mathcal{H}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g;r}$ . The forgetful map  $\phi_{g;r} : \overline{\mathcal{H}}_{g;r} \rightarrow \overline{\mathcal{H}}_g$  is proper, flat and surjective with connected fibers, and thus is a fibration. Let  $\mathcal{H}_{g;r} = \overline{\mathcal{H}}_{g;r} \times_{\overline{\mathcal{H}}_g} \mathcal{H}_g$ .

## 2.2. Stratifications

Let  $X$  be a principally polarized semiabelian variety of dimension  $g$  defined over  $k$ . Its  $p$ -rank  $f(X) = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, X)$  is an integer between 0 and  $g$ . If  $X$  is an abelian variety, then  $X[p](k) \cong (\mathbb{Z}/p)^{f(X)}$ . The  $p$ -rank of a curve is that of its Jacobian. If  $X \rightarrow S$  is a semiabelian scheme over a Deligne–Mumford stack, then there is a stratification  $S = \bigcup S^f$  by locally closed substacks such that  $s \in S^f(k)$  if and only if  $f(X_s) = f$  (this follows from [20, Thm. 2.3.1], see, e.g., [4, Lemma 2.1]). Thus,  $\mathcal{H}_g^f$  is the locus in  $\mathcal{H}_g$  parametrizing hyperelliptic curves of  $p$ -rank  $f$ . Every component of  $\mathcal{H}_g^f$  has dimension  $g-1+f$  [18, Thm. 1].

Here is the definition of the Newton polygon of an abelian variety; see [12, Chap. IV] for details. The isogeny class of a  $p$ -divisible group  $G/k$  is determined by  $v(G)$ , a lower-convex polygon in  $\mathbb{R}^2$  connecting  $(0,0)$  to  $(\mathrm{height}(G), \mathrm{dim}(G))$  with slopes  $\lambda \in \mathbb{Q} \cap [0, 1]$  and integral breakpoints. If  $X/k$  is an abelian variety, its Newton polygon is that of its  $p$ -divisible group  $X[p^\infty]$ . The Newton polygon is a finer invariant than the  $p$ -rank; indeed, the  $p$ -rank of  $X$  is exactly the length of the slope 0 part of the Newton polygon. For example,  $X$  is ordinary exactly when its Newton polygon only has slopes 0 and 1. By definition,  $X$  is supersingular if its Newton polygon only has slope  $1/2$ .

## 2.3. The boundary of the moduli space of hyperelliptic curves

The boundary of  $\overline{\mathcal{H}}_g$  is  $\partial\overline{\mathcal{H}}_g = \overline{\mathcal{H}}_g - \mathcal{H}_g$ . The description here of  $\partial\overline{\mathcal{H}}_g$  follows [10, Sec. 4(b)] closely. In fact, while [10] is written for the base field  $\mathbb{C}$ , the description of  $\mathcal{H}_g$  and  $\overline{\mathcal{H}}_g$  is valid in any characteristic [38]. Briefly, the irreducible components of  $\partial\overline{\mathcal{H}}_g$  come from restriction of the components of the boundary of  $\overline{\mathcal{M}}_g$ , except that  $\Delta_0$  breaks into several components. The informal discussion in this section is supplemented with precise definitions in Section 2.4.

If  $g \geq 2$ , the boundary  $\partial\overline{\mathcal{H}}_g$  is the union of components  $\Delta_i = \Delta_i[\overline{\mathcal{H}}_g]$  for  $1 \leq i \leq g-1$  and  $\mathcal{E}_i = \mathcal{E}_i[\overline{\mathcal{H}}_g]$  for  $0 \leq i \leq g-2$  by [38, p. 410]. Here  $\Delta_i$  and  $\Delta_{g-i}$  are the same substack of  $\overline{\mathcal{H}}_g$  and  $\mathcal{E}_i$  and  $\mathcal{E}_{g-i-1}$  are the same substack of  $\overline{\mathcal{H}}_g$ . Each  $\Delta_i$  and  $\mathcal{E}_i$  is an irreducible divisor in  $\overline{\mathcal{H}}_g$ .

For  $1 \leq i \leq g-1$ , if  $\eta$  is the generic point of  $\Delta_i$ , then the curve  $\mathcal{C}_{g,\eta}$  is a chain of two smooth irreducible hyperelliptic curves  $Y_1$  and  $Y_2$ , of genera  $i$  and  $g-i$ , intersecting in one ordinary double point  $P$ . The hyperelliptic involution  $\iota$  stabilizes each of  $Y_1$  and  $Y_2$ . The point  $P$  is a ramification point for the restriction of  $\iota$  to each of  $Y_1$  and  $Y_2$  but is not part of the smooth ramification locus.

If  $\eta$  is the generic point of  $\mathcal{E}_0$ , then the curve  $\mathcal{C}_{g,\eta}$  is an irreducible hyperelliptic curve self-intersecting in an ordinary double point  $P$ . The normalization of  $\mathcal{C}_{g,\eta}$  is a smooth hyperelliptic curve  $Y_1$  of genus  $g-1$  and the inverse image of  $P$  in the normalization consists of an orbit under the hyperelliptic involution.

For  $1 \leq i \leq g-2$ , if  $\eta$  is the generic point of  $\mathcal{E}_i$ , then the curve  $\mathcal{C}_{g,\eta}$  has two components  $Y_1$  and  $Y_2$ , which are smooth irreducible hyperelliptic curves, of genera  $i$  and  $g-1-i$ , intersecting in two ordinary double points  $P$  and  $Q$ . The hyperelliptic involution  $\iota$  stabilizes each of  $Y_1$  and  $Y_2$ . The points  $P$  and  $Q$  form an orbit of the restriction of  $\iota$  to each of  $Y_1$  and  $Y_2$ .

One can associate to a stable curve  $C$  its dual graph, in which the vertices are in bijection with the irreducible components of  $C$  and in which there is an edge between two vertices exactly when the corresponding components intersect. A component of  $C$  is called terminal if the corresponding vertex is a leaf of the dual graph. A curve is called a tree if its dual graph is a tree. A curve is called a tree of elliptic curves if it is a tree and if each of its irreducible components is an elliptic curve.

A stable curve is a tree if and only if its Picard variety is represented by an abelian scheme; such a curve is also said to be of compact type. Let  $\Delta_0 = \Delta_0[\overline{\mathcal{H}}_g]$  be the union of  $\mathcal{E}_i$  for  $0 \leq i \leq \lfloor (g-1)/2 \rfloor$ . The moduli points of stable hyperelliptic curves which are not of compact type are exactly the points of  $\Delta_0[\overline{\mathcal{H}}_g]$ .

If  $S$  is a stack equipped with a map  $S \rightarrow \overline{\mathcal{H}}_g$ , let  $\Delta_i[S]$  denote  $S \times_{\overline{\mathcal{H}}_g} \Delta_i[\overline{\mathcal{H}}_g]$ . In particular,  $\Delta_i[\tilde{\mathcal{H}}_g] = \tilde{\mathcal{H}}_g \times_{\overline{\mathcal{H}}_g} \Delta_i$ . Also define  $\Delta_i[\overline{\mathcal{H}}_g]^f := (\Delta_i[\overline{\mathcal{H}}_g])^f$ . Similar conventions are employed for  $\mathcal{E}_i$ .

## 2.4. Clutching maps

Recall from [22] that there are three types of clutching maps for positive integers  $g_1$  and  $g_2$ :

$$\kappa_{g_1, g_2} : \tilde{\mathcal{H}}_{g_1} \times \tilde{\mathcal{H}}_{g_2} \xrightarrow{\tilde{\kappa}_{g_1, g_2}} \tilde{\mathcal{H}}_{g_1+g_2} \xrightarrow{\varpi_{g_1+g_2}} \overline{\mathcal{H}}_{g_1+g_2};$$

$$\kappa_{g_1} : \overline{\mathcal{H}}_{g_1;1} \longrightarrow \overline{\mathcal{H}}_{g_1+1};$$

$$\lambda_{g_1, g_2} : \overline{\mathcal{H}}_{g_1;1} \times \overline{\mathcal{H}}_{g_2;1} \longrightarrow \overline{\mathcal{H}}_{g_1+g_2+1}.$$

Each of these clutching maps is the restriction of a finite, unramified morphism between moduli spaces of labeled curves [22, Cor. 3.9]. One defines

$$\Delta_{g_1}[\overline{\mathcal{H}}_{g_1+g_2}] = \text{Im}(\kappa_{g_1, g_2}); \quad \mathcal{E}_0[\overline{\mathcal{H}}_{g_1+1}] = \text{Im}(\kappa_{g_1}); \quad \text{and} \quad \mathcal{E}_{g_1}[\overline{\mathcal{H}}_{g_1+g_2+1}] = \text{Im}(\lambda_{g_1, g_2}).$$

These clutching maps can be described in terms of their images on  $T$ -points for an arbitrary  $k$ -scheme  $T$ . We give explicit descriptions only for sufficiently general  $T$ -points and for the one special case (in 2.4.3) we need later, and defer to [22] for complete definitions.

### 2.4.1. Information about $\kappa_{g_1, g_2}$

For  $i = 1, 2$ , suppose  $s_i \in \tilde{\mathcal{H}}_{g_i}(T)$  is the moduli point of a hyperelliptic curve  $Y_i$  with labeled smooth ramification locus. Then  $\tilde{\kappa}_{g_1, g_2}(s_1, s_2)$  is the moduli point of the labeled  $T$ -curve  $Y$  where  $Y$  has components  $Y_1$  and  $Y_2$ , and where the last ramification point of  $Y_1$  and the first ramification point of  $Y_2$  are identified in an ordinary double point. This nodal section is dropped

from the labeling of the ramification points; the remaining smooth ramification sections of  $Y_1$  maintain labels  $\{1, \dots, 2g_1 + 1\}$  and the remaining ramification sections of  $Y_2$  are relabeled  $\{2g_1 + 2, \dots, 2(g_1 + g_2) + 2\}$ . There is a unique hyperelliptic involution on  $Y$  which restricts to the hyperelliptic involution on  $Y_1$  and  $Y_2$ . Moreover,  $\kappa_{g_1, g_2}(s_1, s_2)$  is the moduli point of the (unlabeled) hyperelliptic curve  $Y$ .

By [7, Ex. 9.2.8],

$$\mathrm{Pic}^0(Y) \cong \mathrm{Pic}^0(Y_1) \times \mathrm{Pic}^0(Y_2). \quad (2.4.1)$$

Then the  $p$ -rank of  $Y$  is

$$f(Y) = f(Y_1) + f(Y_2). \quad (2.4.2)$$

#### 2.4.2. Information about $\kappa_{g_1}$

Suppose  $s_1 \in \overline{\mathcal{H}}_{g_1;1}(T)$  is the moduli point of  $(Y_1; P)$ , a hyperelliptic curve with a labeled section. Consider the  $T$ -curve  $Y'$  obtained by identifying the sections  $P$  and  $\iota(P)$  in an ordinary double point  $P'$ . Then  $\kappa_{g_1}(s_1)$  is the moduli point of the stable model  $Y$  of  $Y'$ . The hyperelliptic involution on  $Y_1$  descends to a unique hyperelliptic involution on  $Y$ .

By [7, Ex. 9.2.8],  $\mathrm{Pic}^0(Y)$  is an extension

$$0 \longrightarrow Z \longrightarrow \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Pic}^0(Y_1) \longrightarrow 0, \quad (2.4.3)$$

where  $Z$  is a one-dimensional torus. In particular, the toric rank of  $\mathrm{Pic}^0(Y)$  is one greater than that of  $\mathrm{Pic}^0(Y_1)$ , and their maximal projective quotients are isomorphic, so that

$$f(Y) = f(Y_1) + 1. \quad (2.4.4)$$

#### 2.4.3. Information about $\lambda_{g_1, g_2}$

For  $i = 1, 2$ , suppose  $s_i \in \overline{\mathcal{H}}_{g_i;1}(T)$  is the moduli point of  $(Y_i; P_i)$ , a hyperelliptic curve with a labeled section. Consider the  $T$ -curve  $Y'$  obtained by identifying  $P_1$  and  $P_2$  in an ordinary double point  $P$  and by identifying  $\iota(P_1)$  and  $\iota(P_2)$  in an ordinary double point  $Q$ . Then  $\lambda_{g_1, g_2}(s_1, s_2)$  is the moduli point of the stable model  $Y$  of  $Y'$ . There is a unique hyperelliptic involution  $\iota_Y$  on  $Y$  which restricts to the hyperelliptic involution on  $Y_1$  and  $Y_2$ .

By [7, Ex. 9.2.8],  $\mathrm{Pic}^0(Y)$  is an extension

$$0 \longrightarrow Z \longrightarrow \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Pic}^0(Y_1) \times \mathrm{Pic}^0(Y_2) \longrightarrow 0, \quad (2.4.5)$$

where  $Z$  is a one-dimensional torus. In particular,

$$f(Y) = f(Y_1) + f(Y_2) + 1. \quad (2.4.6)$$

For later use, here is a description of the stable model  $Y$  when  $P_1$  is a ramification point of  $Y_1$ , but  $P_2$  is not a ramification point of  $Y_2$ . Then  $Y$  consists of three components, namely the strict transforms of  $Y_1$  and  $Y_2$  and an exceptional component  $W$  which is a projective line. Also  $Y_1$  intersects  $W$  in an ordinary double point  $P$  and  $Y_2$  intersects  $W$  in two other ordinary double points,  $Q$  and  $Q'$ . The restriction of  $\iota_Y$  to  $W$  fixes  $P$  and exchanges  $Q$  and  $Q'$ .

#### 2.4.4. Clutching along trees

The definition of  $\tilde{\kappa}_{g_1, g_2}$  above relies on an arbitrary, although convenient, choice of sections along which to glue, and a labeling of the smooth ramification locus of the resulting curve. By considering morphisms of the form  $\gamma_{g_1+g_2} \circ \tilde{\kappa}_{g_1, g_2} \circ (\gamma_{g_1} \times \gamma_{g_2})$ , where  $\gamma_{g_1+g_2} \in \text{Sym}_{2(g_1+g_2)+2}$  and  $\gamma_{g_1} \in \text{Sym}_{2g_1+2}$  and  $\gamma_{g_2} \in \text{Sym}_{2g_2+2}$ , it is possible to clutch along arbitrary sections, with complete control over the subsequent labeling.

This can be used to describe configurations of curves of compact type, as follows. A clutching tree is a finite tree  $\Lambda$  along with a choice of natural number  $g_v$  for each vertex  $v \in \Lambda$  such that  $\deg(v) \leq 2g_v + 2$ . Such a tree is called a clutching tree of elliptic curves if  $g_v = 1$  for all  $v \in \Lambda$ . Let  $|\Lambda|$  be the number of vertices in  $\Lambda$ , and let  $g(\Lambda) = \sum_{v \in \Lambda} g_v$ .

Using a product of the clutching maps defined above, one can define a morphism

$$\kappa_\Lambda : \times_{v \in V} \tilde{\mathcal{H}}_{g_v} \xrightarrow{\tilde{\kappa}_\Lambda} \tilde{\mathcal{H}}_{g(\Lambda)} \xrightarrow{\varpi_{g(\Lambda)}} \overline{\mathcal{H}}_{g(\Lambda)}.$$

Let  $\Delta_\Lambda$  be the image of  $\kappa_\Lambda$ . If  $\eta$  is the generic point of  $\Delta_\Lambda$ , then  $\mathcal{C}_{g, \eta}$  is a hyperelliptic curve of compact type with dual graph isomorphic to  $\Lambda$ , such that the irreducible component of  $\mathcal{C}_{g, \eta}$  corresponding to the vertex  $v$  has genus  $g_v$ .

Suppose  $v_1$  and  $v_2$  are adjacent vertices in a clutching tree  $\Lambda$ . Consider the tree  $\Lambda'$  obtained by identifying  $v_1$  and  $v_2$  in a new vertex,  $v$ , which is adjacent to all neighbors of  $v_1$  or  $v_2$  in  $\Lambda$ , with label  $g_v = g_{v_1} + g_{v_2}$ . Then  $\kappa_\Lambda$  factors through  $\kappa_{\Lambda'}$  and  $\Delta_\Lambda \subset \Delta_{\Lambda'}$ . A tree  $\Lambda$  refines a tree  $\Lambda'$  if  $\Lambda'$  can be obtained from  $\Lambda$  through iterations of this construction.

#### 2.4.5. One more clutching map

In the special case where  $\Lambda$  is a tree on three vertices, where the leaves  $v_1$  and  $v_3$  have  $g_{v_1} = g_{v_3} = 1$ , and where  $g_{v_2} = g - 2$ , one obtains the following diagram:

$$\begin{array}{ccc} \tilde{\mathcal{H}}_1 \times \tilde{\mathcal{H}}_{g-2} \times \tilde{\mathcal{H}}_1 & \longrightarrow & \tilde{\mathcal{H}}_{g-1} \times \tilde{\mathcal{H}}_1 \\ \downarrow & \searrow \tilde{\kappa}_{1, g-2, 1} & \downarrow \\ \tilde{\mathcal{H}}_1 \times \tilde{\mathcal{H}}_{g-1} & \longrightarrow & \tilde{\mathcal{H}}_g. \end{array} \quad (2.4.7)$$

Let  $\Delta_{1,1}[\overline{\mathcal{H}}_g]$  be the image of  $\kappa_{1, g-2, 1} = \varpi_g \circ \tilde{\kappa}_{1, g-2, 1}$ ; it is an irreducible component of the self-intersection locus of  $\Delta_1[\overline{\mathcal{H}}_g]$ . If  $\eta$  is the generic point of  $\Delta_{1,1}[\overline{\mathcal{H}}_g]$ , then the curve  $\mathcal{C}_{g, \eta}$  is a chain of three smooth irreducible hyperelliptic curves  $Y_1, Y_2, Y_3$  with  $g_{Y_1} = g_{Y_3} = 1$  and  $g_{Y_2} = g - 2$ . For  $i \in \{1, 3\}$ , the curves  $Y_i$  and  $Y_2$  intersect in a point  $P_i$  which is an ordinary double point.



### 3. Boundary of the hyperelliptic $p$ -rank strata

#### 3.1. Preliminary intersection results

Suppose  $p \geq 3$ , and  $g \geq 1$  and  $0 \leq f \leq g$ . The  $p$ -rank strata of the boundary of  $\overline{\mathcal{H}}_g$  are easy to describe using the clutching maps. First, if  $1 \leq i \leq g-1$ , then (2.4.2) implies that  $\Delta_i[\overline{\mathcal{H}}_g]^f$  is the union of the images of  $\tilde{\mathcal{H}}_i^{f_1} \times \tilde{\mathcal{H}}_{g-i}^{f_2}$  under  $\kappa_{i,g-i}$  as  $(f_1, f_2)$  ranges over all pairs such that

$$0 \leq f_1 \leq i, \quad 0 \leq f_2 \leq g-i \quad \text{and} \quad f_1 + f_2 = f. \quad (3.1.1)$$

Second, if  $f \geq 1$ , then  $\Xi_0[\overline{\mathcal{H}}_g]^f$  is the image of  $\overline{\mathcal{H}}_{g-1;1}^{f-1}$  under  $\kappa_{g-1}$  by (2.4.4). Third, if  $f \geq 1$  and  $1 \leq i \leq g-2$ , then (2.4.6) implies that  $\Xi_i[\overline{\mathcal{H}}_g]^f$  is the image of  $\overline{\mathcal{H}}_{i;1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i;1}^{f_2}$  under  $\lambda_{i,g-1-i}$  as  $(f_1, f_2)$  ranges over all pairs such that

$$0 \leq f_1 \leq i, \quad 0 \leq f_2 \leq g-1-i \quad \text{and} \quad f_1 + f_2 = f-1. \quad (3.1.2)$$

**Lemma 3.1.** Suppose  $g \geq 1$  and  $0 \leq f \leq g$ .

- (a) Every component of  $\overline{\mathcal{H}}_g^f$  and  $\tilde{\mathcal{H}}_g^f$  has dimension  $g-1+f$ .
- (b) If  $g \geq 2$  and  $1 \leq i \leq g-1$ , then every component of  $\Delta_i[\overline{\mathcal{H}}_g]^f$  and of  $\Delta_i[\tilde{\mathcal{H}}_g]^f$  has dimension  $g-2+f$ .
- (c) If  $g \geq 2$  and  $f \geq 1$  and  $0 \leq i \leq g-2$ , then every component of  $\Xi_i[\overline{\mathcal{H}}_g]^f$  and of  $\Xi_i[\tilde{\mathcal{H}}_g]^f$  has dimension  $g-2+f$ .

**Proof.** For parts (a)–(c), the claims for  $\overline{\mathcal{H}}_g$  and for  $\tilde{\mathcal{H}}_g$  are equivalent, since  $\varpi_g$  is a finite map which preserves the  $p$ -rank stratification. Part (a) for  $\overline{\mathcal{H}}_g^f$  is the statement of [18, Prop. 2].

For part (b), suppose  $0 \leq f \leq g$ ,  $1 \leq i \leq g-1$ , and that  $(f_1, f_2)$  is a pair which satisfies (3.1.1). Then  $\tilde{\mathcal{H}}_i^{f_1} \times \tilde{\mathcal{H}}_{g-i}^{f_2}$  is pure of dimension  $\dim(\tilde{\mathcal{H}}_i^{f_1}) + \dim(\tilde{\mathcal{H}}_{g-i}^{f_2}) = g-2+f$ . Since  $\kappa_{i,g-i}$  is finite,  $\Delta_i[\overline{\mathcal{H}}_g]^f$  is pure of dimension  $g-2+f$  as well.

For part (c), first suppose  $i=0$ . Then  $\overline{\mathcal{H}}_{g-1;1}^{f-1}$  is pure of dimension  $\dim(\overline{\mathcal{H}}_{g-1}^{f-1}) + 1 = g-2+f$ . Since  $\kappa_{g-1}$  is finite,  $\Xi_0[\overline{\mathcal{H}}_g]^f$  is pure of dimension  $g-2+f$  as well.

To finish part (c), suppose  $1 \leq i \leq g-2$ , and that  $(f_1, f_2)$  is a pair which satisfies (3.1.2). Then  $\overline{\mathcal{H}}_{i;1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i;1}^{f_2}$  is pure of dimension  $\dim(\overline{\mathcal{H}}_i^{f_1}) + \dim(\overline{\mathcal{H}}_{g-1-i}^{f_2}) + 2 = g-2+f$ . Since  $\lambda_{i,g-1-i}$  is finite,  $\Xi_i[\overline{\mathcal{H}}_g]^f$  is pure of dimension  $g-2+f$  as well.  $\square$

The next lemma shows that if  $\eta$  is a generic point of  $\overline{\mathcal{H}}_g^f$ , then the curve  $\mathcal{C}_{g,\eta}$  is smooth. Thus no component of  $\overline{\mathcal{H}}_g^f$  is contained in the boundary  $\partial\overline{\mathcal{H}}_g$ .

**Lemma 3.2.** Suppose  $g \geq 1$  and  $0 \leq f \leq g$ .

- (a) Then  $\mathcal{H}_g^f$  is open and dense in  $\overline{\mathcal{H}}_g^f$  and  $\tilde{\mathcal{H}}_g^f \times_{\overline{\mathcal{H}}_g} \mathcal{H}_g$  is open and dense in  $\tilde{\mathcal{H}}_g^f$ .
- (b) If  $r \geq 1$ , then  $\mathcal{H}_{g,r}^f$  is open and dense in  $\overline{\mathcal{H}}_{g,r}^f$  and  $\tilde{\mathcal{H}}_g^f \times_{\overline{\mathcal{H}}_g} \mathcal{H}_g$  is open and dense in  $\tilde{\mathcal{H}}_g^f$ .

**Proof.** Part (a) is well known for  $g = 1$ . For  $g \geq 2$ , part (a) follows from Lemma 3.1. Part (b) follows from part (a) since the  $p$ -rank of a labeled curve depends only on the underlying curve, so that  $\overline{\mathcal{H}}_{g;r}^f = \overline{\mathcal{H}}_{g;r} \times_{\overline{\mathcal{H}}_g} \overline{\mathcal{H}}_g^f$ .  $\square$

Suppose  $S$  is an irreducible component of  $\mathcal{H}_g^f$  and let  $\bar{S}$  be its closure in  $\overline{\mathcal{H}}_g$ . Note that  $\bar{S}$  can contain the moduli points of curves with lower  $p$ -rank; (in fact, this always happens when  $f \geq 1$ , see Corollary 3.15).

**Lemma 3.3.**

- (a) Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ . If  $\bar{S}$  intersects a component  $\Gamma$  of  $\Delta_i[\overline{\mathcal{H}}_g]^f$  then  $\bar{S}$  contains  $\Gamma$ .
- (b) Let  $\tilde{S}$  be an irreducible component of  $\tilde{\mathcal{H}}_g^f$ , with closure  $S^*$ . If  $S^*$  intersects a component  $\tilde{\Gamma}$  of  $\Delta_i[\tilde{\mathcal{H}}_g]^f$ , then  $S^*$  contains  $\tilde{\Gamma}$ .

**Proof.** Part (a) is proved here; part (b) is proved in an entirely analogous fashion. A smooth proper stack has the same intersection-theoretic properties as a smooth proper scheme [36, p. 614]. In particular, if two closed substacks of  $\overline{\mathcal{H}}_g$  intersect then the codimension of their intersection is at most the sum of their codimensions. Now  $\bar{S}$  and  $\Delta_i[\overline{\mathcal{H}}_g]$  are closed substacks of  $\overline{\mathcal{H}}_g$  and  $\bar{S} \not\subset \Delta_i[\overline{\mathcal{H}}_g]$ . Thus the intersection of  $\bar{S}$  with the divisor  $\Delta_i[\overline{\mathcal{H}}_g]$  has pure dimension  $\dim \bar{S} - 1$ , which equals  $\dim(\Delta_i[\overline{\mathcal{H}}_g]^f)$  by Lemma 3.1. Thus if  $\bar{S}$  intersects a component  $\Gamma$  of  $\Delta_i[\overline{\mathcal{H}}_g]^f$  then it must contain the full component  $\Gamma$ .  $\square$

**Lemma 3.4.** Suppose  $g \geq 2$  and  $0 \leq f \leq g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ .

- (a) Then  $\bar{S}$  intersects  $\Delta_0[\overline{\mathcal{H}}_g]$  if and only if  $f \geq 1$ .
- (b) If  $f \geq 1$ , then each irreducible component of  $\Delta_0[\bar{S}]$  contains either (i) the image of a component of  $\overline{\mathcal{H}}_{g-1;1}^{f-1}$  under  $\kappa_{g-1}$  or (ii) the image of a component of  $\overline{\mathcal{H}}_{i;1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i;1}^{f_2}$  under  $\lambda_{i,g-1-i}$  for some  $1 \leq i \leq g-2$  and some pair  $(f_1, f_2)$  which satisfies (3.1.2).
- (c) If  $f = 0$ , then  $\bar{S}$  contains the image of a component of  $\tilde{\mathcal{H}}_i^0 \times \tilde{\mathcal{H}}_{g-i}^0$  under  $\kappa_{i,g-i}$  for some  $1 \leq i \leq g-1$ .

**Proof.** If  $f = 0$ , then (2.4.4) implies that  $\bar{S}$  does not intersect  $\Delta_0[\overline{\mathcal{H}}_g]$ . If  $f \geq 1$ , then  $\bar{S}$  is a complete substack of dimension greater than  $g-1$ . By [16, Lemma 2.6], a complete substack of  $\overline{\mathcal{H}}_g - \Delta_0[\overline{\mathcal{H}}_g]$  has dimension at most  $g-1$ . Therefore,  $\bar{S}$  intersects  $\Delta_0[\overline{\mathcal{H}}_g]$ . This completes part (a).

For part (b), each irreducible component of  $\Delta_0[\bar{S}]$  intersects either  $\kappa_{g-1}(\overline{\mathcal{H}}_{g-1;1}^{f-1}) \subset \mathcal{E}_0[\overline{\mathcal{H}}_g]$  or  $\lambda_{i,g-1-i}(\overline{\mathcal{H}}_{i;1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i;1}^{f_2}) \subset \mathcal{E}_i[\overline{\mathcal{H}}_g]$  for some  $1 \leq i \leq g-2$  and some pair  $(f_1, f_2)$  which satisfies (3.1.2). The result then follows from Lemma 3.3(a).

For part (c), recall that  $\mathcal{H}_g$  contains no complete substacks of positive dimension (e.g., [38, Cor. 1.9]). Thus  $\bar{S}$  intersects  $\Delta_i$  for some  $0 \leq i \leq g-1$ . By part (a),  $i \neq 0$ . The result follows from Lemma 3.3(a).  $\square$

### 3.2. Complements on trees

Many of the results for  $\Delta_i$  for positive  $i$  have analogues for  $\Delta_\Lambda$ . For a clutching tree  $\Lambda$  and a nonnegative integer  $f$ , define an index set by

$$\mathcal{F}(\Lambda, f) = \left\{ \{f_v : v \in \Lambda\} : 0 \leq f_v \leq g_v, \sum_v f_v = f \right\}. \quad (3.2.1)$$

**Lemma 3.5.** *Let  $\Lambda$  be a clutching tree with  $g(\Lambda) = g$ .*

(a) *The  $p$ -rank strata of  $\Delta_\Lambda[\overline{\mathcal{H}}_g]$  are given by*

$$\Delta_\Lambda[\overline{\mathcal{H}}_g]^f = \bigcup_{\{f_v\} \in \mathcal{F}(\Lambda, f)} \kappa_\Lambda(\times_{v \in \Lambda} \widetilde{\mathcal{H}}_{g_v}^{f_v}). \quad (3.2.2)$$

(b) *Every component of  $\Delta_\Lambda[\overline{\mathcal{H}}_g]^f$  has dimension  $g + f - |\Lambda|$ .*

**Proof.** Part (a) follows from (3.1.1) and induction on  $|\Lambda|$ . Part (b) follows from this and the calculation that, for  $\{f_v\} \in \mathcal{F}(\Lambda, f)$ ,

$$\dim(\times_{v \in \Lambda} \widetilde{\mathcal{H}}_{g_v}^{f_v}) = \sum_{v \in \Lambda} (g_v + f_v - 1) = g + f - |\Lambda|. \quad \square$$

**Lemma 3.6.** *Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ . Let  $\Lambda$  be a clutching tree with  $g(\Lambda) = g$ . If  $\bar{S}$  intersects a component  $\Gamma$  of  $\Delta_\Lambda[\overline{\mathcal{H}}_g]^f$ , then  $\bar{S}$  contains  $\Gamma$ .*

**Proof.** The proof is similar to that of Lemma 3.3(a). Note that  $\dim \Gamma \geq \dim \bar{S} + \dim \Delta_\Lambda - \dim \overline{\mathcal{H}}_g$ . By Lemma 3.5(b), this equals  $g + f - |\Lambda| = \dim \Delta_\Lambda[\overline{\mathcal{H}}_g]^f$ .  $\square$

### 3.3. Adjusting labeled points and trees

The next lemma shows that one can adjust the labeled points of an  $r$ -labeled hyperelliptic curve of genus  $g$  and  $p$ -rank  $f$  without leaving the irreducible component of  $\overline{\mathcal{H}}_{g;r}^f$  to which its moduli point belongs.

**Lemma 3.7.** *Let  $S$  be an irreducible component of  $\mathcal{H}_{g;r}^f$ , and let  $\bar{S}$  be the closure of  $S$  in  $\overline{\mathcal{H}}_{g;r}^f$ . Then  $\bar{S} = \phi_{g;r}^{-1}(\phi_{g;r}(\bar{S}))$ . Equivalently, if  $T$  is a  $k$ -scheme, if  $(C; P_1, \dots, P_r) \in \bar{S}(T)$ , and if  $(Q_1, \dots, Q_r)$  is any other labeling of  $C$ , then  $(C; Q_1, \dots, Q_r) \in \bar{S}(T)$ .*

**Proof.** It suffices to show that  $\phi_{g;r}^{-1}(\phi_{g;r}(\bar{S})) \subseteq \bar{S}$ . Note that  $\bar{S}$  is the largest irreducible substack of  $\overline{\mathcal{H}}_{g;r}^f$  which contains  $S$ . The fibers of  $\phi_{g;r}|_S$  are irreducible, so  $\phi_{g;r}^{-1}(\phi_{g;r}(S))$  is also an irreducible substack of  $\overline{\mathcal{H}}_{g;r}^f$  which contains  $S$ . Thus  $\phi_{g;r}^{-1}(\phi_{g;r}(S)) \subset \bar{S}$ . This shows that  $\phi_{g;r}^{-1}(\phi_{g;r}(S)) = S$ .

To finish the proof, it suffices to show that the  $T$ -points of  $\bar{S}$  and  $\phi_{g;r}^{-1}(\phi_{g;r}(\bar{S}))$  coincide for an arbitrary  $k$ -scheme  $T$ . To this end, let  $\alpha = (C; P_1, \dots, P_r) \in \bar{S}(T)$ , and let  $\beta = (C; Q_1, \dots, Q_r) \in \bar{\mathcal{H}}_{g;r}^f(T)$ . Note that  $\phi_{g;r}(\beta) = \phi_{g;r}(\alpha)$ , and  $\phi_{g;r}(\alpha)$  is supported in the closure of  $\phi_{g;r}(S)$  in  $\bar{\mathcal{H}}_g^f$ . Because  $\mathcal{H}_{g;r}$  is dense in  $\bar{\mathcal{H}}_{g;r}$ , it follows that  $\beta$  is supported in the closure of  $\phi_{g;r}^{-1}(\phi_{g;r}(S))$  in  $\bar{\mathcal{H}}_{g;r}^f$ , which is  $\bar{S}$ .  $\square$

It is not clear whether one can change the labeling of the smooth ramification locus of a hyperelliptic curve without changing the irreducible component of  $\tilde{\mathcal{H}}_g^f$  to which its moduli point belongs. To circumvent this issue, the following lemma about hyperelliptic curves of genus 2 and  $p$ -rank 1 will be useful.

**Lemma 3.8.**

- (a) First,  $\bar{\mathcal{H}}_2^1$  is irreducible and intersects  $\kappa_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$ .
- (b) Second, let  $\tilde{S}$  be an irreducible component of  $\tilde{\mathcal{H}}_2^1$ . If  $\tilde{S}$  intersects  $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$ , then  $\tilde{S}$  also intersects  $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_1^1)$ .

**Proof.** For part (a), recall that the Torelli morphism  $\mathcal{H}_2 \rightarrow \mathcal{A}_2$  is an inclusion [33, Lemma 1.11]. Since  $\dim(\mathcal{H}_2^1) = \dim(\mathcal{A}_2^1)$ , and since  $\mathcal{A}_2^1$  is irreducible (e.g., [15, Ex. 11.6]), it follows that  $\bar{\mathcal{H}}_2^1$  is irreducible. Consider a chain  $Y$  of two elliptic curves, one ordinary and one supersingular, intersecting in an ordinary double point, which is a fixed point of the hyperelliptic involution on each elliptic curve. The moduli point of  $Y$  is in the intersection of  $\kappa_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$  and  $\bar{\mathcal{H}}_2^1$ .

For part (b), let  $S^*$  be the closure of  $\tilde{S}$  in  $\tilde{\mathcal{H}}_2$ . By hypothesis and Lemma 3.3(b),  $\tilde{S}$  contains a component of  $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$ . Every component of  $\tilde{\mathcal{H}}_1^1$  contains a component of  $\tilde{\mathcal{H}}_1^0$  in its closure since any nonisotrivial proper family of curves of genus one has supersingular fibers. It follows that  $S^*$  contains a component of  $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_1^0)$ , and thus intersects the closure of a component of  $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_1^1)$ . By Lemma 3.3(b),  $S^*$  contains a component of  $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^1)$ , which then implies the same for  $\tilde{S}$ .  $\square$

**Remark 3.9.** Note that a genus two curve has six ramification points and thus there are potentially up to  $6! = |\text{Sym}_6|$  irreducible components of  $\tilde{\mathcal{H}}_2^1$ . In particular, the fact from Lemma 3.8(a) that  $\bar{\mathcal{H}}_2^1$  intersects  $\kappa_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$  does not imply the hypothesis in part (b) that  $\tilde{S}$  intersects  $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$ .

**Lemma 3.10.** Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ . Suppose  $\Lambda$  is a clutching tree of elliptic curves with  $g(\Lambda) = g$ . If  $\bar{S}$  intersects  $\Delta_\Lambda[\bar{\mathcal{H}}_g]$ , then for any choice of  $\{f_v\} \in \mathcal{F}(\Lambda, f)$ ,  $\bar{S}$  contains an irreducible component of  $\kappa_\Lambda(\times_{v \in \Lambda} \tilde{\mathcal{H}}_{g_v}^{f_v})$ .

**Proof.** By Lemma 3.6, there exist a choice of data  $\{f_v^*\} \in \mathcal{F}(\Lambda, f)$  and components  $\tilde{T}_v \subset \tilde{\mathcal{H}}_{g_v}^{f_v^*}$  such that  $\bar{S}$  contains  $\kappa_\Lambda(\times_{v \in \Lambda} \tilde{T}_v)$ . One immediately reduces to the case in which  $v_1$  and  $v_2$  are adjacent vertices in  $\Lambda$  with  $f_{v_1}^* = 1$  and  $f_{v_2}^* = 0$ , and  $\{f_v\} \in \mathcal{F}(\Lambda, f)$  is given by

$$f_v = \begin{cases} f_v^*, & v \notin \{v_1, v_2\}, \\ 1 - f_v^*, & v \in \{v_1, v_2\}. \end{cases}$$

Let  $\Lambda'$  be the tree obtained by identifying  $v_1$  and  $v_2$  in a new vertex  $v_{12}$  with  $g_{v_{12}} = 2$ . By Lemma 3.6, there is a component  $\tilde{T}_{v_{12}} \subset \tilde{\mathcal{H}}_2^1$  such that  $\bar{S}$  contains

$$\kappa_{\Lambda'}(\tilde{T}_{v_{12}} \times (\times_{v \in \Lambda', v \neq v_{12}} \tilde{T}_v)).$$

Now,  $\tilde{T}_{v_{12}}$  contains a component of  $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \mathcal{H}_1^0)$ . By Lemma 3.8(b),  $\tilde{T}_{v_{12}}$  contains a component of  $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_1^1)$  as well. Then  $\bar{S}$  contains a component of  $\kappa_{\Lambda}(\times_{v \in \Lambda} \tilde{\mathcal{H}}_{g_v}^{f_v})$ .  $\square$

### 3.4. Main intersection result

In this section, we prove that the closure of each irreducible component  $S$  of  $\mathcal{H}_g^f$  contains the moduli point of a singular curve which is a tree of elliptic curves and has  $p$ -rank  $f$ .

**Theorem 3.11.** *Suppose  $g \geq 2$  and  $0 \leq f \leq g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ .*

- (a) *There exists a clutching tree of elliptic curves  $\Lambda$  with  $g(\Lambda) = g$  such that  $\bar{S}$  contains an irreducible component of  $\Delta_{\Lambda}[\tilde{\mathcal{H}}_g]^f$ .*
- (b) *For any choice of  $\{f_v\} \in \mathcal{F}(\Lambda, f)$ ,  $\bar{S}$  contains an irreducible component of  $\kappa_{\Lambda}(\times_{v \in \Lambda} \tilde{\mathcal{H}}_{g_v}^{f_v})$ .*
- (c) *In particular,  $\bar{S}$  contains the moduli point of some tree of elliptic curves, of which  $f$  are ordinary and  $g - f$  are supersingular.*

**Proof.** It suffices to prove part (c) since parts (a) and (c) are equivalent by Lemma 3.6 and since parts (a) and (b) are equivalent by Lemma 3.10.

First suppose  $g = 2$ . If  $f = 2$ , then  $\mathcal{H}_2^2$  is irreducible and affine and  $\bar{\mathcal{H}}_2^2$  contains the moduli point of a tree of 2 ordinary elliptic curves. If  $f = 1$  (resp.  $f = 0$ ), the result is true by Lemma 3.8(a) (resp. Lemma 3.4(c)). Now suppose  $g \geq 3$  and  $0 \leq f \leq g$  and suppose as an inductive hypothesis that the result is true when  $2 \leq g' < g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ .

**Claim 3.12.** *To complete the proof, it suffices to show that  $\bar{S}$  intersects  $\Delta_i[\bar{\mathcal{H}}_g]^f$  for some  $1 \leq i \leq g - 1$ .*

**Proof of claim.** Suppose  $\bar{S}$  intersects  $\Delta_i[\bar{\mathcal{H}}_g]^f$  for some  $1 \leq i \leq g - 1$ . Let  $g_1 = i$  and  $g_2 = g - i$ . By Lemma 3.3(a),  $\bar{S}$  contains a component of  $\Delta_{g_1}[\bar{\mathcal{H}}_{g_1}]^f$ . In other words, there exist a pair  $(f_1, f_2)$  satisfying (3.1.1) and, for  $j = 1, 2$ , components  $\tilde{V}_j$  of  $\tilde{\mathcal{H}}_{g_j}^{f_j}$  such that  $\bar{S}$  contains  $\kappa_{g_1, g_2}(\tilde{V}_1 \times \tilde{V}_2)$ . Then  $\bar{V}_j = \varpi_{g_j}(\tilde{V}_j)$  is a component of  $\bar{\mathcal{H}}_{g_j}^{f_j}$ .

By the inductive hypothesis,  $\bar{V}_j$  contains the moduli point  $s_j$  of a tree  $Y_j$  of  $g_j$  elliptic curves, of which  $f_j$  are ordinary and  $g_j - f_j$  are supersingular. Let  $\Lambda_j$  be the dual graph of  $Y_j$ . Let  $\tilde{s}_j \in \tilde{V}_j$  be such that  $\varpi_{g_j}(\tilde{s}_j) = s_j$ . In other words,  $\tilde{s}_j$  is the moduli point of  $Y_j$  along with the data of a choice of labeling of the smooth ramification locus. Then  $\kappa_{g_1, g_2}(\tilde{s}_1, \tilde{s}_2)$  is the moduli point of a curve  $C$  whose dual graph is obtained by connecting a vertex of  $\Lambda_1$  with a vertex of  $\Lambda_2$ . Since  $C$  is a tree,  $\kappa_{g_1, g_2}(\tilde{s}_1, \tilde{s}_2)$  is the moduli point of a tree of  $g$  elliptic curves, of which  $f = f_1 + f_2$  are ordinary and  $g - f$  are supersingular. This completes the proof of the claim since  $\kappa_{g_1, g_2}(\tilde{s}_1, \tilde{s}_2)$  is in  $\bar{S}$ .  $\square$

Continuing the proof of Theorem 3.11, first suppose  $f = 0$ . By Lemma 3.4(c),  $\bar{S}$  intersects  $\Delta_i[\bar{\mathcal{H}}_g]^f$  for some  $1 \leq i \leq g - 1$ . By Claim 3.12, this completes the proof when  $f = 0$ .

Now suppose  $f > 0$ . By Lemma 3.4(a),  $\bar{S}$  intersects  $\Delta_0[\bar{\mathcal{H}}_g]^f$ .

**Case (i):  $\bar{S}$  intersects  $\mathcal{E}_0$ .**

By Lemma 3.4(b),  $\bar{S}$  contains the image of a component  $\bar{V}'$  of  $\bar{\mathcal{H}}_{g-1;1}^{f-1}$  under  $\kappa_{g-1}$ . Consider  $\bar{V} = \phi_{g-1;1}(\bar{V}')$  which is a component of  $\bar{\mathcal{H}}_{g-1}^{f-1}$ . By the inductive hypothesis,  $\bar{V}$  contains the moduli point of a curve  $Y_1$  which is a tree of  $g - 1$  elliptic curves, of which  $f - 1$  are ordinary and  $g - f$  are supersingular. Let  $E$  be a terminal component of  $Y_1$  and let  $Y'_1$  be the closure of  $Y_1 - E$  in  $Y_1$ . Let  $R$  be the point of intersection of  $E$  and  $Y'_1$ . Since the quotient of  $Y_1$  by the hyperelliptic involution  $\iota$  has genus 0, the elliptic curve  $E$  is stabilized by  $\iota$ . Let  $P \neq R$  be a point of  $E$  which is not a ramification point of  $\iota$ . By Lemma 3.7, the moduli point  $t$  of  $(Y_1; P)$  is in  $\bar{V}'$ .

Let  $Z$  be the singular irreducible hyperelliptic curve of genus two with exactly one ordinary double point  $P'$  such that the normalization of  $Z$  is the elliptic curve  $E$  and the pre-image of  $P'$  consists of the points  $P$  and  $\iota(P)$ . In other words, the moduli point of  $Z$  is the image of the moduli point of  $(E; P)$  under  $\kappa_1$ .

Consider the point  $s = \kappa_{g-1}(t)$  of  $\bar{S}$ . The curve  $\mathcal{C}_{g,s}$  has components  $Z$  and  $Y'_1$  which intersect in exactly one ordinary double point  $R$ . The  $p$ -rank of  $\mathcal{C}_{g,s}$  is  $f(E) + f(Y'_1) + 1 = f$ . Since  $g \geq 3$ , there is a terminal component of  $Y'_1$  not containing  $R$  which is an elliptic curve. Thus  $s$  is in  $\Delta_1[\bar{\mathcal{H}}_g]^f$ . (In fact,  $s$  is also in  $\Delta_2[\bar{\mathcal{H}}_g]^f$  because of the component  $Z$ .) By Claim 3.12, this completes case (i).

**Case (ii):  $\bar{S}$  intersects  $\mathcal{E}_i$  for some  $1 \leq i \leq g - 2$ .**

Let  $g_1 = i$  and  $g_2 = g - 1 - i$ . By Lemma 3.4(b),  $\bar{S}$  contains a component of  $\mathcal{E}_i[\bar{\mathcal{H}}_g]^f$ . In other words, there exists a pair  $(f_1, f_2)$  satisfying (3.1.2) and, for  $j = 1, 2$ , there exist components  $\bar{V}'_j$  of  $\bar{\mathcal{H}}_{g_j;1}^{f_j}$  such that  $\bar{S}$  contains  $\lambda_{g_1,g_2}(\bar{V}'_1 \times \bar{V}'_2)$ . Then  $\bar{V}_j = \phi_{g_j;1}(\bar{V}'_j)$  is a component of  $\bar{\mathcal{H}}_{g_j}^{f_j}$ .

By the inductive hypothesis,  $\bar{V}_j$  contains the moduli point  $s_j$  of a tree  $Y_j$  of  $g_j$  elliptic curves, of which  $f_j$  are ordinary and  $g_j - f_j$  are supersingular. Let  $E_j$  be a terminal component of  $Y_j$  and let  $Y'_j$  be the closure of  $Y_j - E_j$  in  $Y_j$ . Let  $R_j$  be the point of intersection of  $E_j$  and  $Y'_j$ . Since the quotient of  $Y_j$  by the hyperelliptic involution  $\iota_j$  has genus 0, the elliptic curve  $E_j$  is stabilized by  $\iota_j$ . Let  $P_1 \neq R_1$  be a point of  $E_1$  which is a ramification point of  $\iota_1$ . Let  $P_2 \neq R_2$  be a point of  $E_2$  which is not a ramification point of  $\iota_2$ .

By Lemma 3.7, the moduli point  $s'_j$  of  $(Y_j; P_j)$  is in  $\bar{V}'_j$ . Consider  $s = \lambda_{g_1,g_2}(s'_1, s'_2)$  which is a point of  $\bar{S}$ . By Section 2.4.3, the components of the stable model  $\mathcal{C}_{g,s}$  are the strict transforms of  $Y_1$  and  $Y_2$  and an exceptional component  $W$  which is a projective line. Moreover,  $Y_1$  intersects  $W$  in an ordinary double point and  $Y_2$  intersects  $W$  in two other points, which are also ordinary double points. The  $p$ -rank of  $\mathcal{C}_{g,s}$  is  $f(Y_1) + f(Y_2) + 1 = f$ .

The curve  $\mathcal{C}_{g,s}$  has a terminal component  $E'_1$  of genus 1. To see this, when  $i = 1$  let  $E'_1 = E_1$ , and when  $i > 1$  let  $E'_1 \neq E_1$  be another terminal component of  $Y_1$ . It follows that  $s$  is in  $\Delta_1[\bar{\mathcal{H}}_g]^f$ . (Also  $s$  is in  $\Delta_i[\bar{\mathcal{H}}_g]^f$  because of the component  $Y_1$ .) By Claim 3.12, this completes case (ii).  $\square$

### 3.5. Three corollaries

Here are several consequences of Theorem 3.11 which will be used later in the paper.

In the setting of Theorem 3.11, one can deduce that  $\bar{S}$  intersects  $\Delta_i$  nontrivially only when  $\Lambda$  has an edge whose removal yields two trees of size  $i$  and  $g - i$ . This is only guaranteed

when  $i = 1$ . Luckily, the following information on degeneration to  $\Delta_1$  is sufficient for the later applications in the paper.

**Corollary 3.13.** *Suppose  $g \geq 2$  and  $0 \leq f \leq g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ . Then  $\bar{S}$  intersects  $\Delta_1[\tilde{\mathcal{H}}_g]^f$ . Furthermore:*

- (a) *if  $f \leq g - 1$ , then  $\bar{S}$  contains an irreducible component of  $\kappa_{1,g-1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_{g-1}^f)$ ; and*
- (b) *if  $f \geq 1$ , then  $\bar{S}$  contains an irreducible component of  $\kappa_{1,g-1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_{g-1}^{f-1})$ .*

**Proof.** By Theorem 3.11(c),  $\bar{S}$  contains the moduli point of a tree of elliptic curves, of which  $f$  are ordinary and  $g - f$  are supersingular. Every tree has a leaf; by Theorem 3.11(b), that leaf can be chosen to be ordinary or supersingular if the obvious necessary constraint is satisfied. The result follows by Lemma 3.6.  $\square$

The  $\ell$ -adic and  $p$ -adic monodromy proofs in Section 5 rely on degeneration to  $\Delta_{1,1}$ . One can label the four possibilities for  $(f_1, f_2, f_3)$  such that  $f_1 + f_2 + f_3 = f$  and  $0 \leq f_1, f_3 \leq 1$  as follows: (A)  $(1, f - 2, 1)$ ; (B)  $(0, f - 1, 1)$ ; (B')  $(1, f - 1, 0)$ ; and (C)  $(0, f, 0)$ .

**Corollary 3.14.** *Suppose  $g \geq 3$  and  $0 \leq f \leq g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ .*

- (a) *Then  $\bar{S}$  intersects  $\Delta_{1,1}[\tilde{\mathcal{H}}_g]^f$ .*
- (b) *There is an irreducible component  $\tilde{S}$  of  $\bar{S} \times \tilde{\mathcal{H}}_g$ , and a choice of  $(f_1, f_2, f_3)$  from cases (A)–(C); and there are irreducible components  $S_1$  of  $\tilde{\mathcal{H}}_1^{f_1}$  and  $S_2$  of  $\tilde{\mathcal{H}}_{g-2}^{f_2}$  and  $S_3$  of  $\tilde{\mathcal{H}}_1^{f_3}$ ; and there are irreducible components  $S_R$  of  $\tilde{\mathcal{H}}_{g-1}^{f_2+f_3}$  and  $S_L$  of  $\tilde{\mathcal{H}}_{g-1}^{f_1+f_2}$ ; such that the restriction of the clutching maps of 2.4.7 yields a commutative diagram*

$$\begin{array}{ccc}
 S_1 \times S_2 \times S_3 & \xrightarrow{\quad} & S_1 \times S_R \\
 \downarrow & \searrow \tilde{\kappa}_{1,g-2,1} & \downarrow \\
 S_L \times S_3 & \xrightarrow{\quad} & \tilde{S} \cap \Delta_{1,1}[\tilde{\mathcal{H}}_g]^f.
 \end{array} \tag{3.5.1}$$

- (c) *Furthermore, case (A) occurs as long as  $f \geq 2$ , case (B) or (B') occurs as long as  $1 \leq f \leq g - 1$ , and case (C) occurs as long as  $f \leq g - 2$ .*

**Proof.** By Theorem 3.11(a), there is a clutching tree of elliptic curves  $\Lambda$  such that  $\bar{S}$  contains a component of  $\Delta_\Lambda[\tilde{\mathcal{H}}_g]^f$ . Let  $v_1$  and  $v_3$  be two leaves of  $\Lambda$ ; using Theorem 3.11(b), one can assume that  $f_{v_i} = f_i$  where  $(f_1, f_2, f_3)$  is chosen as in part (c). Let  $\Lambda'$  be the tree obtained by coalescing all vertices of  $\Lambda$  except for  $v_1$  and  $v_3$ . Let  $v_2$  denote this new vertex and let  $f_{v_2} = f_2$ . Since  $\Lambda$  refines  $\Lambda'$ , then  $\bar{S}$  intersects  $\Delta_\Lambda[\tilde{\mathcal{H}}_g]^f$  which completes part (a). Moreover, there is an irreducible component  $\tilde{S}$  of  $\bar{S} \times_{\tilde{\mathcal{H}}_g} \tilde{\mathcal{H}}_g$  such that  $\tilde{S}$  intersects  $\Delta_{1,1}[\tilde{\mathcal{H}}_g]^f$ . Part (b) follows from the definition of  $\Delta_{1,1}[\tilde{\mathcal{H}}_g]^f$  and Lemma 3.6.  $\square$

**Corollary 3.15.** *Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ . For each  $0 \leq f' < f$ , there exists an irreducible component  $T$  of  $\mathcal{H}_g^{f'}$  such that  $\bar{S}$  contains  $T$ .*

**Proof.** It suffices to prove the result for  $f' = f - 1$ . Let  $S^*$  be the closure of  $S$  in  $\overline{\mathcal{H}}_g - \Delta_0[\overline{\mathcal{H}}_g]$ . A purity result [29, Lemma 1.6] shows that  $S^* - (S^*)^f$ , if nonempty, is pure of dimension  $\dim S^* - 1$ . In particular, let  $Z = (S^*)^{f-1}$ ; then  $Z$ , if nonempty, is pure of dimension  $g - 2 + f$ .

By Corollary 3.13(b),  $S^*$  contains an irreducible component of  $\kappa_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_{g-1}^{f-1})$ . Since  $\tilde{\mathcal{H}}_1^1$  is dense in  $\tilde{\mathcal{H}}_1$ , its closure contains the moduli points of supersingular elliptic curves (with labeled smooth ramification locus). Therefore,  $S^*$  contains an irreducible component of  $\kappa_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_{g-1}^{f-1})$ , and  $Z$  is nonempty. Then  $\dim Z = g - 2 + f = \dim \overline{\mathcal{H}}_g^{f-1}$ , and so  $Z$  contains a component  $\bar{T}$  of  $\overline{\mathcal{H}}_g^{f-1}$ . By Lemma 3.2(a)  $\bar{S}$  contains a component  $T$  of  $\mathcal{H}_g^{f-1}$ .  $\square$

### 3.6. Application to Newton polygons

Recall that a stable curve  $C$  of compact type is supersingular if all the slopes of the Newton polygon of its Jacobian equal  $1/2$ . This is equivalent to the condition that the Jacobian of  $C$  is isogenous to a product of supersingular elliptic curves. Note that a supersingular curve necessarily has  $p$ -rank zero. An abelian variety of  $p$ -rank zero is necessarily supersingular only when the dimension satisfies  $g \leq 2$ .

In this section, we prove that the Newton polygon of a generic hyperelliptic curve of  $p$ -rank 0 is not supersingular when  $g \geq 3$ . The result generalizes [31, Thm. 1.12] which is the case  $g = 3$ .

Newton polygons have the following semicontinuity property: let  $S = \operatorname{Spec}(R)$  be the spectrum of a local ring, with generic point  $\eta$  and geometric closed point  $s$ ; if  $G$  is a  $p$ -divisible group over  $S$ , then  $\nu(G_\eta)$  either equals or lies below  $\nu(G_s)$ . (The latter condition means that  $\nu(G_\eta)$  and  $\nu(G_s)$  have the same endpoints and all points of  $\nu(G_\eta)$  lie below  $\nu(G_s)$ .)

**Corollary 3.16.** *Suppose  $p$  is an odd prime and  $g \geq 3$ . Let  $\eta$  be a generic point of  $\mathcal{H}_g^0$ . Then  $\mathcal{C}_{g,\eta}$  is not supersingular. In particular, there exists a smooth hyperelliptic curve of genus  $g$  and  $p$ -rank 0 which is not supersingular.*

**Proof.** When  $g = 3$ , this follows from [31, Thm. 1.12]. For  $g \geq 4$ , the proof proceeds by induction. Let  $S$  be the closure of  $\eta$  in  $\mathcal{H}_g^0$ . By Corollary 3.13,  $\bar{S}$  contains a component of  $\Delta_1[\overline{\mathcal{H}}_g]^0$ . Thus there are components  $\tilde{V}_1$  of  $\tilde{\mathcal{H}}_1^0$  and  $\tilde{V}_2$  of  $\tilde{\mathcal{H}}_{g-1}^0$  such that  $\bar{S}$  contains  $\kappa_{1,g-1}(\tilde{V}_1 \times \tilde{V}_2)$ . By the inductive hypothesis, the Newton polygon of the generic point of  $\tilde{V}_2$  is not supersingular; in particular, it has a slope  $\lambda$  such that  $0 < \lambda < 1/2$ . The same is then true of the generic point of  $\kappa_{1,g-1}(\tilde{V}_1 \times \tilde{V}_2)$ . By semicontinuity [20, Thm. 2.3.1], the generic Newton polygon of  $\bar{S}$  (and thus of  $S$ ) either equals or lies below that of  $\kappa_{1,g-1}(\tilde{V}_1 \times \tilde{V}_2)$ . In particular, it has a slope  $\lambda' < 1/2$ . Thus  $\mathcal{C}_{g,\eta}$  is not supersingular.  $\square$

**Remark 3.17.** When  $p = 2$  (a case not considered in this paper), there are some results about the slopes of Newton polygons of hyperelliptic curves of  $p$ -rank 0, see e.g. [6,34].

### 3.7. Open questions about the geometry of the hyperelliptic $p$ -rank strata

**Question 3.18.** Does the closure of each component of  $\mathcal{H}_g^f$  contain the moduli point of a chain of elliptic curves with  $p$ -rank  $f$ ?

More generally, one could ask which trees arise in the boundary of  $\mathcal{H}_g^f$ . If the answer to Question 3.18 is affirmative then Lemma 3.10 implies that every ordering of  $f$  ordinary and  $g - f$  supersingular elliptic curves occurs for such a chain. In [4, Cor. 3.6], the authors show the



analogous question has a positive answer for every component of  $\mathcal{M}_g^f$ . The difference for  $\mathcal{M}_g^f$  is that the clutching morphism identifies two curves at an arbitrary point of each, rather than a ramification point of each. The location of these points can then be changed using an analogue of Lemma 3.7.

**Question 3.19.** For  $2 \leq i \leq g-2$ , does the closure of each component of  $\mathcal{H}_g^f$  intersect  $\Delta_i[\mathcal{H}_g]^f$ ?

In [4, Prop. 3.4], the authors show that the analogous question has a positive answer for every component of  $\mathcal{M}_g^f$ , also with control over the arrangement of  $p$ -ranks. An affirmative answer to Question 3.18 would imply an affirmative answer to Question 3.19.

**Question 3.20.** How many irreducible components does  $\mathcal{H}_g^f$  have?

One knows that  $\mathcal{H}_g^f$  is irreducible for all  $p$  when  $f = g$  or when  $g = 2$  and  $f = 1$ . If  $g \geq 3$ , then  $\mathcal{A}_g^f$  is irreducible by [8, Remark 4.7]. If  $\mathcal{H}_g^f$  is irreducible, then there is a very short proof that Questions 3.18 and 3.19 have affirmative answers.

**Question 3.21.** Let  $\eta$  be a generic point of  $\mathcal{H}_g^f$ . What is the Newton polygon of the Jacobian of  $\mathcal{C}_{g,\eta}$ ?

The answer to Question 3.21 is known only for pairs  $(g, f)$  such that  $g - f \leq 2$ . For the generic point  $\xi$  of  $\mathcal{A}_g^0$  with  $g \geq 3$ , one knows that the Newton polygon of the abelian variety  $X_\xi$  is  $\{1/g, (g-1)/g\}$  [32, Cor. 3.2].

#### 4. Endomorphism rings

In this section, we use degeneration to constrain the endomorphism ring of a generic curve of given genus and  $p$ -rank.

Let  $\mathcal{X}_g = \text{Pic}_{\mathcal{C}_g/\overline{\mathcal{H}}_g}^0$  be the neutral component of the relative Picard functor of  $\mathcal{C}_g$  over  $\overline{\mathcal{H}}_g$ ; then  $\mathcal{X}_g \rightarrow \overline{\mathcal{H}}_g$  is a semiabelian scheme. To ease notation, if  $X$  is an abelian variety, let  $\mathbf{E}(X) = \text{End}(X) \otimes \mathbb{Q}$ , and let  $\mathbf{E}_\ell(X) = \text{End}(X) \otimes \mathbb{Z}_\ell$ ; then  $\mathbf{E}_\ell(X)$  acts on the Tate module  $T_\ell(X)$ , and the endomorphism algebra  $\mathbf{E}(X)$  acts on each rational Tate module  $V_\ell(X) := T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ . If  $X$  is simple, then the center of  $\mathbf{E}(X)$  is either a totally real or totally imaginary number field.

**Lemma 4.1.** *Let  $\xi$  be a geometric generic point of  $\mathcal{H}_3^0$ . Then  $\mathcal{X}_{3,\xi}$  is simple and either  $\mathbf{E}(\mathcal{X}_{3,\xi}) \cong \mathbb{Q}$  or  $\mathbf{E}(\mathcal{X}_{3,\xi})$  is a totally real cubic field.*

**Proof.** Suppose there is an isogeny  $\mathcal{X}_{3,\xi} \sim A_1 \oplus A_2$  for abelian varieties  $A_1$  and  $A_2$  of dimensions 1 and 2. Then  $A_1$  and  $A_2$  each have  $p$ -rank 0 and are thus supersingular. Then  $\mathcal{X}_{3,\xi}$  is supersingular, which contradicts the fact that the Newton polygon of  $\mathcal{X}_{3,\xi}$  has slopes  $1/3$  and  $2/3$  [31, Thm. 1.12]. Thus  $\mathcal{X}_{3,\xi}$  is simple.

By the classification of endomorphism algebras of simple abelian varieties of prime dimension (e.g., [30, 7.2]), to complete the proof it suffices to show that neither a complex multiplication field of degree six nor a quadratic imaginary field acts on  $\mathcal{X}_{3,\xi}$ . Let  $S$  be the closure of  $\xi$  in  $\mathcal{H}_3^0$ . Since  $\dim S = 2 > 0$  but abelian varieties with complex multiplication are rigid,  $\mathbf{E}(\mathcal{X}_{3,\xi})$  is not a complex multiplication field of degree 6.

To address the possibility of an action by a quadratic imaginary field  $K$ , suppose to the contrary that there is a subring of  $\text{End}(\mathcal{X}_{3,\xi})$  isomorphic to an order  $\mathcal{O}_K$  in  $K$ . Then  $\mathcal{O}_K \otimes \mathbb{Z}_p$  acts on the  $p$ -divisible group  $\mathcal{X}_{3,\xi}[p^\infty]$ . There is an inclusion  $K \otimes \mathbb{Q}_p \hookrightarrow \text{End}(\mathcal{X}_{3,\xi}[p^\infty]) \otimes \mathbb{Q}_p \cong D_{1/3} \oplus D_{2/3}$ . (Here,  $D_\lambda$  denotes the central simple  $\mathbb{Q}_p$ -algebra with Brauer invariant  $\lambda$ .) Every maximal subfield of  $D_{1/3}$  or  $D_{2/3}$  is a cubic extension of  $\mathbb{Q}_p$ , but  $K \otimes \mathbb{Q}_p$  is a  $\mathbb{Q}_p$ -algebra of degree two, so  $K \otimes \mathbb{Q}_p$  cannot be a field. In particular,  $\mathcal{X}_{3,\xi}$  does not admit an action by a quadratic imaginary field inert or ramified at  $p$ .

Finally, suppose  $\mathcal{X}_{3,\xi}$  admits an action by a quadratic imaginary field  $K$  which splits at  $p$ . Let  $(r, s)$  be the signature of the action of  $\mathcal{O}_K$  on  $\text{Lie}(\mathcal{X}_{3,\xi})$ ; the dimensions  $r$  and  $s$  are nonnegative and  $r + s = 3$ . Consider the moduli space  $Sh_{\mathcal{O}_K;(r,s)}$  of abelian threefolds with an action by  $\mathcal{O}_K$  of signature  $(r, s)$ . The Torelli morphism  $\tau$  restricts to a finite morphism from  $S$  to a component of  $Sh_{\mathcal{O}_K;(r,s)}$ . Since  $\dim S = 2$  and  $\dim Sh_{\mathcal{O}_K;(r,s)} = r \cdot s$ , then  $(r, s)$  is either  $(1, 2)$  or  $(2, 1)$ . Thus  $\tau(S)$  is dense in  $Sh_{\mathcal{O}_K;(r,s)}$ . This gives a contradiction since  $\mathcal{X}_{3,\xi}$  has  $p$ -rank zero but the generic member of  $Sh_{\mathcal{O}_K;(r,s)}$  is ordinary [37, Thm. 1.6.2].  $\square$

**Question 4.2.** Is  $\text{End}(\text{Pic}^0(C_{3,\xi})) \cong \mathbb{Z}$  for every geometric generic point  $\xi$  of  $\mathcal{H}_3^0$ ?

Consider the situation of Lemma 4.1. Using the description of the  $p$ -rank strata of Hilbert modular threefolds, one can show that if  $\mathbf{E}(\mathcal{X}_{3,\xi})$  is a cubic field  $L$ , then  $L$  is either (totally) inert or ramified at  $p$ . The authors tentatively expect that  $\mathbf{E}(\mathcal{X}_{3,\xi})$  is always  $\mathbb{Q}$ , simply because there is no obvious source of such a totally real cubic field for each  $p$ . Note that since the number of irreducible components of  $\mathcal{H}_3^0$  remains unknown (see Question 3.20), even numerical experiments remain challenging.

**Lemma 4.3.** *Let  $Y$  be a simple abelian variety whose dimension  $g$  is relatively prime to 3. If there exists a geometric generic point  $\xi_3$  of  $\mathcal{H}_3^0$  for which there is a nontrivial homomorphism  $\text{End}(Y) \rightarrow \mathbf{E}(\mathcal{X}_{3,\xi_3})$ , then  $\text{End}(Y) \cong \mathbb{Z}$ .*

**Proof.** If  $Z$  is a simple abelian variety, let  $\mathbf{E}_0(Z)$  be the subfield of  $\mathbf{E}(Z)$  fixed by the Rosati involution, and let  $e_0(Z) = [\mathbf{E}_0(Z) : \mathbb{Q}]$ . Then  $e_0(Z) \mid \dim(Z)$ .

By Lemma 4.1,  $\mathbf{E}(\mathcal{X}_{3,\xi_3})$  is a totally real field of dimension 1 or 3 over  $\mathbb{Q}$ . On one hand, the existence of a nontrivial homomorphism  $\text{End}(Y) \rightarrow \mathbf{E}(\mathcal{X}_{3,\xi_3})$  forces  $e_0(Y)$  to divide  $e_0(\mathcal{X}_{3,\xi_3})$ , and thus  $e_0(Y) \mid 3$ . On the other hand,  $e_0(Y) \mid g$ . Therefore,  $e_0(Y) = 1$  and  $e_0(Y) \cong \mathbb{Q}$ . Neither a noncommutative algebra nor a totally imaginary field admits a nontrivial homomorphism to  $\mathbf{E}(\mathcal{X}_{3,\xi_3})$ , and thus  $\mathbf{E}(Y) = \mathbf{E}_0(Y)$  and  $\text{End}(Y) \cong \mathbb{Z}$ .  $\square$

**Lemma 4.4.** *Let  $X \rightarrow S$  be a polarized abelian scheme over a reduced irreducible Noetherian stack. Let  $\eta$  be the generic point of  $S$ , and let  $s \in S$  be any point. Then there exists an inclusion  $\text{End}(X_\eta) \hookrightarrow \text{End}(X_s)$ .*

**Proof.** By introducing a rigidifying structure on  $X \rightarrow S$ , such as coordinates on the space of sections of the third power of the ample line bundle given by the polarization, one can assume  $S$  is a reduced irreducible Noetherian scheme. Since the absolute endomorphism ring of an abelian variety is defined over a finite extension of the base field, it suffices to show the existence of an inclusion  $\text{End}(X_\eta) \hookrightarrow \text{End}(X_s)$ . If  $S$  is normal, then  $\text{End}(X_\eta)$  extends uniquely to  $\text{End}(X_s)$ , and in particular to  $\text{End}(X_s)$  [17, I.2.7]. In general, let  $\nu : S' \rightarrow S$  be the normalization map; let  $\eta'$  be the generic point of  $S'$ , and let  $s'$  be a point of  $S'$  over  $s$ . The desired result follows from

the canonical map  $\text{End}((v^*X)_{\eta'}) \hookrightarrow \text{End}((v^*X)_{s'})$  and the isomorphisms of abelian varieties  $(v^*X)_{\eta'} \cong X_{\eta} \times \eta'$  and  $(v^*X)_{s'} \cong X_s \times s'$ .  $\square$

**Proposition 4.5.** *If  $\xi_4$  is a geometric generic point of  $\mathcal{H}_4^0$ , then  $\mathcal{X}_{4,\xi_4}$  is simple and  $\text{End}(\mathcal{X}_{4,\xi_4}) \cong \mathbb{Z}$ .*

**Proof.** Let  $S_4$  be the closure of  $\xi_4$  in  $\mathcal{H}_4^0$ . Suppose there is an isogeny  $\mathcal{X}_{4,\xi_4} \sim A_1 \oplus A_2$  for two abelian varieties  $A_1$  and  $A_2$ . If  $A_1$  and  $A_2$  each have dimension 2, then they are supersingular since they have  $p$ -rank 0. Then  $\mathcal{X}_{4,\xi_4}$  is supersingular, which contradicts Corollary 3.16. If  $A_1$  has dimension 1 and  $A_2$  has dimension 3, then there is a curve  $W$  of genus 3 such that  $\text{Jac}(W) \cong A_2$ . The inclusion of  $A_2$  into  $\mathcal{X}_{4,\xi_4}$  yields a cover  $\psi: \mathcal{C}_{4,\xi_4} \rightarrow W$ . By the Riemann–Hurwitz formula  $6 \geq 4 \deg(\psi)$  which is impossible since  $\deg(\psi) \geq 2$ . Thus  $\mathcal{X}_{4,\xi_4}$  is simple.

By Corollary 3.13(a) there exist components  $\tilde{V}_1 \subset \tilde{\mathcal{H}}_1^0$  and  $\tilde{V}_2 \subset \tilde{\mathcal{H}}_3^0$  such that  $\bar{S}_4$  contains  $\kappa_{1,3}(\tilde{V}_1 \times \tilde{V}_2)$ . Let  $\xi_1$  and  $\xi_3$  be geometric generic points of  $\tilde{V}_1$  and  $\tilde{V}_2$ , respectively, and let  $\eta = \kappa_{1,3}(\xi_1, \xi_3)$ . Since  $\mathcal{X}_{3,\xi_3}$  is simple by Lemma 4.1, there are no nontrivial homomorphisms between  $\mathcal{X}_{3,\xi_3}$  and  $\mathcal{X}_{1,\xi_1}$ . This yields an isomorphism

$$\mathbf{E}(\mathcal{X}_{4,\eta}) \cong \mathbf{E}(\mathcal{X}_{1,\xi_1}) \oplus \mathbf{E}(\mathcal{X}_{3,\xi_3}).$$

By Lemma 4.4, there is an inclusion  $\mathbf{E}(\mathcal{X}_{4,\xi_4}) \hookrightarrow \mathbf{E}(\mathcal{X}_{4,\eta})$  and thus an inclusion  $\mathbf{E}(\mathcal{X}_{4,\xi_4}) \hookrightarrow \mathbf{E}(\mathcal{X}_{3,\xi_3})$ . Since  $\mathcal{X}_{4,\xi_4}$  is simple, Lemma 4.3 implies that  $\text{End}(\mathcal{X}_{4,\xi_4}) \cong \mathbb{Z}$ .  $\square$

**Theorem 4.6.** *Suppose  $g \geq 4$  and  $0 \leq f \leq g$ . If  $\xi$  is a geometric generic point of  $\mathcal{H}_g^f$ , then  $\text{End}(\mathcal{X}_{g,\xi}) \cong \mathbb{Z}$  and thus  $\mathcal{X}_{g,\xi}$  is simple.*

**Proof.** By Corollary 3.15 and Lemma 4.4, it suffices to prove the result when  $f = 0$ . For  $f = 0$ , the proof is by induction on  $g$  with the base case  $g = 4$  supplied by Proposition 4.5. Suppose  $g \geq 5$  and let  $S_g$  be the closure of  $\xi$  in  $\mathcal{H}_g^0$ . By Corollary 3.14,  $\bar{S}_g$  intersects  $\Delta_{1,1}[\bar{\mathcal{H}}_g]^0$  and there is an irreducible component  $\tilde{S}$  of  $\bar{S}_g \times \tilde{\mathcal{H}}_g$ , and there are irreducible components  $S_1$  of  $\tilde{\mathcal{H}}_1^0$  and  $S_2$  of  $\tilde{\mathcal{H}}_{g-2}^0$  and  $S_3$  of  $\tilde{\mathcal{H}}_1^0$ ; and there are irreducible components  $S_R$  of  $\tilde{\mathcal{H}}_{g-1}^0$  and  $S_L$  of  $\tilde{\mathcal{H}}_{g-1}^0$ ; such that the restriction of the clutching maps yields a commutative diagram

$$\begin{array}{ccc} S_1 \times S_2 \times S_3 & \xrightarrow{\quad} & S_1 \times S_R \\ \downarrow & \searrow \tilde{\kappa}_{1,g-2,1} & \downarrow \\ S_L \times S_3 & \xrightarrow{\quad} & \tilde{S} \cap \Delta_{1,1}[\tilde{\mathcal{H}}_g]. \end{array} \quad (4.0.1)$$

Let  $\eta_i$  be the generic point of  $S_i$  for  $1 \leq i \leq 3$ ; similarly, let  $\eta_L$  be the generic point of  $S_L$ , and  $\eta_R$  that of  $S_R$ . Let  $s = \tilde{\kappa}_{1,g-2,1}(\eta_1, \eta_2, \eta_3)$ . By Lemma 4.4, there are inclusions

$$\begin{array}{ccc} \text{End}(\mathcal{X}_{g,s}) & \longleftarrow & \text{End}(\mathcal{X}_{g,\tilde{\kappa}_{1,g-1}(\eta_1 \times \eta_R)}) \\ \uparrow & & \uparrow \\ \text{End}(\mathcal{X}_{g,\tilde{\kappa}_{g-1,1}(\eta_L \times \eta_3)}) & \longleftarrow & \text{End}(\mathcal{X}_{g,\xi}). \end{array} \quad (4.0.2)$$

Let  $\ell$  be a prime such that  $\mathbf{E}_\ell(E) \cong \mathrm{Mat}_2(\mathbb{Z}_\ell)$  for any supersingular elliptic curve  $E/k$ . There is a canonical isomorphism  $T_\ell(\mathcal{X}_{g,s}) \cong T_\ell(\mathcal{X}_{1,\eta_1}) \times T_\ell(\mathcal{X}_{g-2,\eta_2}) \times T_\ell(\mathcal{X}_{1,\eta_3})$  of  $\mathbb{Z}_\ell$ -modules. Choose coordinates on  $T_\ell(\mathcal{X}_{g,s})$  compatible with this decomposition. By the inductive hypothesis,  $\mathbf{E}(\mathcal{X}_{g-1,\eta_L}) \cong \mathbb{Q}$ , and so  $\mathbf{E}(\mathcal{X}_{g,\tilde{\kappa}_{g-1,1}(\eta_L \times \eta_3)}) \cong \mathbf{E}(\mathcal{X}_{g-1,\eta_L}) \oplus \mathbf{E}(\mathcal{X}_{1,\eta_3}) \cong \mathbb{Q} \oplus \mathbf{E}(\mathcal{X}_{1,\eta_3})$ . Thus  $\mathbf{E}_\ell(\mathcal{X}_{g,\tilde{\kappa}_{g-1,1}(\eta_L \times \eta_3)})$  acts on  $T_\ell(\mathcal{X}_{g,s})$  as  $\mathrm{diag}_{2g-2}(\mathbb{Z}_\ell) \oplus \mathrm{Mat}_2(\mathbb{Z}_\ell)$ . Similarly,  $\mathbf{E}_\ell(\mathcal{X}_{g,\tilde{\kappa}_{1,g-1}(\eta_1 \times \eta_R)})$  acts as  $\mathrm{Mat}_2(\mathbb{Z}_\ell) \oplus \mathrm{diag}_{2g-2}(\mathbb{Z}_\ell)$ . Then

$$\mathbf{E}_\ell(\mathcal{X}_{g,\xi}) \subseteq \mathbf{E}_\ell(\mathcal{X}_{g,\tilde{\kappa}_{g-1,1}(\eta_L \times \eta_3)}) \cap \mathbf{E}_\ell(\mathcal{X}_{g,\tilde{\kappa}_{1,g-1}(\eta_1 \times \eta_R)})$$

so  $\mathbf{E}_\ell(\mathcal{X}_{g,\xi})$  acts on  $T_\ell(\mathcal{X}_{g,s})$  as  $\mathrm{diag}_{2g}(\mathbb{Z}_\ell)$ . Thus,  $\mathbf{E}_\ell(\mathcal{X}_{g,\xi}) \cong \mathbb{Z}_\ell$  and  $\mathrm{End}(\mathcal{X}_{g,\xi}) \cong \mathbb{Z}$ .  $\square$

**Remark 4.7.** For  $g \leq 3$  and  $1 \leq f \leq 3$ , it is also true that  $\mathrm{End}(\mathcal{X}_{3,\xi_3^f}) \cong \mathbb{Z}$  and  $\mathcal{X}_{3,\xi_3^f}$  is simple. More generally, for  $g \geq 2$  and  $f \geq 1$ , Theorem 4.6 can be proved as an application of Theorem 5.2 as in [4, Application 5.7].

## 5. Monodromy

In this section, we determine the  $\ell$ -adic monodromy of components of  $\mathcal{H}_g^f$  for odd primes  $\ell$ . The proof uses an inductive process which depends on the degeneration results from Section 3. For  $f \geq 1$ , the base case  $g = 2$  relies on a special case of [8, Prop. 4.4]. When  $f = 0$ , the base case  $g = 4$  relies on the results on endomorphism rings from Section 4.

### 5.1. Integral monodromy

We summarize the discussion in [3, Sec. 3.1] about  $\mathbb{Z}/\ell$ - and  $\mathbb{Z}_\ell$ -monodromy. Let  $S$  be a connected  $k$ -scheme on which the prime  $\ell$  is invertible. Let  $\pi : C \rightarrow S$  be a relative curve of compact type whose fibres have genus  $g$ . Then  $R^1\pi_*(\mu_\ell)$ , or equivalently  $\mathrm{Pic}^0(C)[\ell]$ , is an étale sheaf of  $\mathbb{Z}/\ell$ -modules. If  $s$  is a geometric point of  $S$ , then  $R^1\pi_*(\mu_\ell)$  is equivalent to a linear representation

$$\rho_{C \rightarrow S, \mathbb{Z}/\ell} : \pi_1(S, s) \rightarrow \mathrm{Aut}((R^1\pi_*(\mu_\ell))_s) \cong \mathrm{GL}_{2g}(\mathbb{Z}/\ell).$$

Let  $M_\ell(C \rightarrow S, s)$  be the image of  $\rho_{C \rightarrow S, \mathbb{Z}/\ell}$  and let  $M_\ell(C \rightarrow S)$  be the isomorphism class of this image as an abstract group. If the family  $C \rightarrow S$  is clear from context, these will be denoted  $M_\ell(S, s)$  and  $M_\ell(S)$ , respectively. There is a canonical polarization on  $\mathrm{Pic}^0(C)$ , and thus (after a choice of  $\ell^{\mathrm{th}}$  root of unity on  $S$ ) there is a symplectic pairing on  $\mathrm{Pic}^0(C)[\ell]$ . Therefore, there is an inclusion of groups  $M_\ell(S) \subseteq \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ . Similarly, for each natural number  $n$  there is a representation

$$\rho_{C \rightarrow S, \mathbb{Z}/\ell^n} : \pi_1(S, s) \rightarrow \mathrm{Aut}((R^1\pi_*(\mu_{\ell^n}))_s).$$

Let  $M_{\mathbb{Z}_\ell}(C \rightarrow S, s) = \varprojlim_n \rho_{C \rightarrow S, \mathbb{Z}/\ell^n}(\pi_1(S, s))$ , and let  $M_{\mathbb{Q}_\ell}(C \rightarrow S, s)$  be the Zariski closure of  $M_{\mathbb{Z}_\ell}(C \rightarrow S, s)$  in  $\mathrm{Aut}(\varprojlim_n (R^1\pi_*(\mu_{\ell^n}))_s \otimes \mathbb{Q}_\ell) \cong \mathrm{GL}_{2g, \mathbb{Q}_\ell}$ .

If  $C \rightarrow S$  is a stable curve such that  $U$ , the locus where  $C$  has compact type, is open and dense, for each coefficient ring  $A \in \{\mathbb{Z}/\ell, \mathbb{Z}_\ell, \mathbb{Q}_\ell\}$  let  $M_A(C \rightarrow S, s) = M_A(C|_U \rightarrow U, s)$ .

One can employ an analogous formalism to define the monodromy group of a relative curve over a stack [27].

## 5.2. Monodromy of the hyperelliptic $p$ -rank strata

In this section, let  $p$  and  $\ell$  be distinct odd primes. We find the integral monodromy of the  $p$ -rank strata  $\mathcal{H}_g^f$  when  $1 \leq f \leq g$ . The integral monodromy of  $\mathcal{H}_g$ , which is the same as the case  $f = g$ , already appears in [3, Thm. 3.4] (see also unpublished work of J.-K. Yu, and [19, Thm. 5.1]).

The following argument shows that to determine the monodromy of families of hyperelliptic curves, one may work with either  $\overline{\mathcal{H}}_g$  or  $\widetilde{\mathcal{H}}_g$ .

**Lemma 5.1.** *Let  $p$  and  $\ell$  be distinct odd primes and suppose  $g \geq 2$ . Let  $S \subset \mathcal{H}_g$  be irreducible and let  $\widetilde{S}$  be an irreducible component of  $\overline{S} \times_{\overline{\mathcal{H}}_g} \widetilde{\mathcal{H}}_g$ . Then  $M_\ell(\widetilde{S}) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$  if and only if  $M_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ .*

**Proof.** Since  $\widetilde{S} \rightarrow S$  is finite,  $M_\ell(\widetilde{S})$  is a subgroup of  $M_\ell(\overline{S})$ . If  $M_\ell(\widetilde{S}) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$  then  $M_\ell(\overline{S})$  is maximal, and thus so are  $M_\ell(\overline{S})$  and  $M_\ell(S)$ .

Conversely, suppose  $M_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ . Since  $\varpi_g$  is étale with Galois group  $\mathrm{Sym}(2g+2)$ , the cover  $\widetilde{S} \rightarrow \overline{S}$  is Galois with Galois group  $G \subseteq \mathrm{Sym}(2g+2)$ . To show  $M_\ell(\widetilde{S}) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ , it suffices by the argument of [3, Lemma 3.3] to show that  $G$  and  $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$  have no common nontrivial quotient. Recall that  $|\mathrm{PSp}_{2g}(\mathbb{Z}/\ell)| = \frac{1}{2} \prod_{1 \leq j \leq g} \ell^{2j-1} (\ell^{2j} - 1)$ . On one hand,  $\mathrm{ord}_\ell |\mathrm{PSp}_{2g}(\mathbb{Z}/\ell)| = g^2$ . On the other hand, for any integer  $N$  we have the estimate  $\mathrm{ord}_\ell(N!) < \sum_{e \geq 1} \frac{N}{\ell^e} = \frac{N}{\ell-1}$ . We thus have  $\mathrm{ord}_\ell(|\mathrm{Sym}(2g+2)|) < \mathrm{ord}_\ell(|\mathrm{PSp}_{2g}(\mathbb{Z}/\ell)|)$  for all  $(g, \ell)$  with  $g \geq 2$  and  $\ell \geq 3$ . Therefore  $\mathrm{PSp}_{2g}(\mathbb{Z}/\ell)$  is not a subquotient of  $\mathrm{Sym}(2g+2)$ , and in particular  $M_\ell(\widetilde{S}) = M_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ .  $\square$

**Theorem 5.2.** *Let  $p$  and  $\ell$  be distinct odd primes. Suppose  $g \geq 1$  and  $1 \leq f \leq g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ , the  $p$ -rank  $f$  stratum in  $\mathcal{H}_g$ . Then  $M_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$  and  $M_{\mathbb{Z}_\ell}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ .*

**Proof.** The proof is by induction on  $g$ . The base cases involve the monodromy of  $\mathcal{H}_2^2$  and  $\mathcal{H}_2^1$ , which follow from [8, Prop. 4.4]; see [4, Thm. 4.5].

Now suppose  $g \geq 3$  and  $1 \leq f \leq g$ . As an inductive hypothesis assume, for all pairs  $(g', f')$  where  $1 \leq f' \leq g' < g$ , that  $M_\ell(S') \cong \mathrm{Sp}_{2g'}(\mathbb{Z}/\ell)$  for every irreducible component  $S'$  of  $\mathcal{H}_{g'}^{f'}$ .

Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ . Recall the degeneration types identified immediately before Corollary 3.14. If  $f = g$ , let  $(f_1, f_2, f_3) = (1, g-2, 1)$  as in case (A); if  $f = g-1$ , let  $(f_1, f_2, f_3) = (0, g-2, 1)$  as in case (B); and if  $1 \leq f \leq g-2$ , let  $(f_1, f_2, f_3) = (0, f, 0)$  as in case (C). By Corollary 3.14, there are irreducible components  $\widetilde{S}$  of  $\overline{S} \times_{\overline{\mathcal{H}}_g} \widetilde{\mathcal{H}}_g$ ,  $\widetilde{S}_1$  of  $\widetilde{\mathcal{H}}_1^{f_1}$ ,  $\widetilde{S}_2$  of  $\widetilde{\mathcal{H}}_{g-2}^{f_2}$  and  $\widetilde{S}_3$  of  $\widetilde{\mathcal{H}}_1^{f_3}$ ; and there are irreducible components  $\widetilde{S}_R$  of  $\widetilde{\mathcal{H}}_{g-1}^{f_2+f_3}$  and  $\widetilde{S}_L$  of  $\widetilde{\mathcal{H}}_{g-1}^{f_1+f_2}$ ; such that the restriction of the clutching maps yields a commutative diagram

$$\begin{array}{ccc}
 \widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3 & \xrightarrow{\quad} & \widetilde{S}_1 \times \widetilde{S}_R \\
 \downarrow & \searrow \scriptstyle \widetilde{\kappa}_{1,g-2,1} & \downarrow \\
 \widetilde{S}_L \times \widetilde{S}_3 & \xrightarrow{\quad} & \Delta_{1,1}[\widetilde{S}].
 \end{array} \tag{5.2.1}$$

In all cases  $f_1 + f_2$  and  $f_2 + f_3$  are positive, and so  $\tilde{S}_L$  and  $\tilde{S}_R$  have monodromy  $\mathrm{Sp}_{2(g-1)}(\mathbb{Z}/\ell)$ , by induction and Lemma 5.1.

The rest of the proof is identical to that of [3, Thm. 3.4]. Briefly, one calculates the monodromy group of  $\tilde{S}$  at a point  $s$  in the image of  $\tilde{S}_1 \times \tilde{S}_2 \times \tilde{S}_3$  under  $\tilde{\kappa}_{1,g-2,1}$ . On one hand, there is an a priori inclusion  $M_\ell(\tilde{S}, s) \subseteq \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ . On the other hand, the previous paragraph shows that  $M_\ell(\tilde{S}, s)$  contains two distinct subgroups isomorphic to  $\mathrm{Sp}_{2(g-1)}(\mathbb{Z}/\ell)$ . A group-theoretic result shows that  $M_\ell(\tilde{S}, s) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ . The result then follows from Lemma 5.1.

The proof that  $M_{\mathbb{Z}_\ell}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$  is identical.  $\square$

### 5.2.1. A $p$ -adic complement

In this section we determine the  $p$ -adic monodromy of components of the  $p$ -rank strata  $\mathcal{H}_g^f$  when  $f \geq 1$ .

Let  $S$  be a connected scheme of characteristic  $p$  with geometric point  $s$ , and let  $X \rightarrow S$  be an abelian scheme with constant  $p$ -rank  $f$ . The group scheme  $X[p]$  and  $p$ -divisible group  $X[p^\infty]$  admit largest étale quotients,  $X[p]^{\mathrm{ét}}$  and  $X[p^\infty]^{\mathrm{ét}}$ . These are respectively classified by homomorphisms  $\pi_1(S, s) \rightarrow \mathrm{Aut}(X[p]^{\mathrm{ét}})_s \cong \mathrm{GL}_f(\mathbb{Z}/p)$  and  $\pi_1(S, s) \rightarrow \mathrm{Aut}(X[p^\infty]^{\mathrm{ét}})_s \cong \mathrm{GL}_f(\mathbb{Z}_p)$ , whose images are denoted  $M_p(X \rightarrow S)$  and  $M_{\mathbb{Z}_p}(X \rightarrow S)$ , or simply  $M_p(S)$  and  $M_{\mathbb{Z}_p}(S)$ .

**Lemma 5.3.** *Suppose  $g \geq 1$  and  $1 \leq f \leq g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ , and let  $\tilde{S}$  be an irreducible component of  $S \times_{\overline{\mathcal{H}}_g} \tilde{\mathcal{H}}_g$ . Then  $M_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$  if and only if  $M_p(\tilde{S}) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ .*

**Proof.** Since  $\tilde{S} \rightarrow S$  is finite,  $M_p(\tilde{S})$  is a subgroup of  $M_p(S)$ . If  $M_p(\tilde{S}) \cong \mathrm{GL}_f(\mathbb{Z}/p)$  then  $M_p(\tilde{S})$  is maximal, and thus so are  $M_p(\tilde{S})$  and  $M_p(S)$ .

Conversely, suppose  $M_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ . Let  $S^*$  be the closure of  $S$  in  $\overline{\mathcal{H}}_g - \Delta_0[\overline{\mathcal{H}}_g]$  and let  $\tilde{S}^*$  be the closure of  $\tilde{S}$  in  $\tilde{\mathcal{H}}_g - \Delta_0[\tilde{\mathcal{H}}_g]$ . Let  $T = S^* - (S^*)^f$  be the locus with  $p$ -rank smaller than  $f$ . Then  $T$  is nonempty by Corollary 3.15. On one hand,  $\tilde{S}^* \rightarrow S^*$  is étale since  $p$  is odd and the cover  $\tilde{S}^* \rightarrow S^*$  is tantamount to a partial level-two structure. On the other hand, the  $\mathrm{GL}_f(\mathbb{Z}/p)$ -cover  $H_f := \mathrm{Hom}_S((\mathbb{Z}/p), \mathrm{Jac}(\mathcal{C}_{g,S})[p]^{\mathrm{ét}}) \rightarrow S$  is ramified along  $T$ . Therefore, the covers  $H_f \rightarrow S$  and  $\tilde{S} \rightarrow S$  are disjoint, and  $M_p(\tilde{S}) = M_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ .  $\square$

**Proposition 5.4.** *Suppose  $g \geq 2$  and  $1 \leq f \leq g$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ . Then  $M_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$  and  $M_{\mathbb{Z}_p}(S) \cong \mathrm{GL}_f(\mathbb{Z}_p)$ .*

**Proof.** First suppose  $f = g$ . When  $g = 2$ , the result for  $\mathcal{H}_2^2$ , or equivalently  $\mathcal{M}_2^2$ , is a special case of [13, Thm. 2.1]. For  $g \geq 3$ , suppose as an inductive hypothesis that  $M_p(\mathcal{H}_{g-1}^{g-1}) \cong \mathrm{GL}_{g-1}(\mathbb{Z}/p)$ . By Lemma 5.3,  $M_p(\tilde{\mathcal{H}}_{g-1}^{g-1}) \cong \mathrm{GL}_{g-1}(\mathbb{Z}/p)$ .

Recall the diagram (2.4.7), and consider a geometric point  $s \in \tilde{\mathcal{H}}_g^g$  in the image of  $\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_{g-2}^{g-2} \times \tilde{\mathcal{H}}_1^1$  under  $\tilde{\kappa}_{1,g-2,1}$ . By the inductive hypothesis and Lemma 5.3,  $M_p(\tilde{\mathcal{H}}_g^g, s)$  contains two distinct copies of  $\mathrm{GL}_{g-1}(\mathbb{Z}/p)$  and thus equals  $\mathrm{GL}_g(\mathbb{Z}/p)$  by the argument of [3, Lemma 3.2].

Now suppose  $1 \leq f \leq g - 1$ . By Corollary 3.13(a), there are irreducible components  $\tilde{V}_1 \subset \tilde{\mathcal{H}}_1^0$  and  $\tilde{V}_2 \subset \tilde{\mathcal{H}}_{g-1}^f$  such that  $\tilde{S}$  contains  $\tilde{\kappa}_{1,g-1}(\tilde{V}_1 \times \tilde{V}_2)$ . By the inductive hypothesis and Lemma 5.3,  $M_p(\tilde{V}_2) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ , and thus  $M_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$  as well.

The proof that  $M_{\mathbb{Z}_p}(S) \cong \mathrm{GL}_f(\mathbb{Z}_p)$  is identical.  $\square$

### 5.3. Monodromy in $p$ -rank zero

In this section, we determine the integral monodromy of components of  $\mathcal{H}_g^0$  under a few mild hypotheses. The monodromy group of  $\mathcal{H}_2^0$  is small, since supersingular families of abelian varieties have finite  $\ell$ -adic monodromy groups; and the methods of [8] do not apply to  $\mathcal{H}_g^0$  for  $g \geq 3$ , because the hyperelliptic Torelli locus is not a Hecke-stable subset of  $\mathcal{A}_g$ . Thus our proof requires another base case when  $f = 0$ . For lack of a strategy to calculate the  $\ell$ -adic monodromy of  $\mathcal{H}_3^0$ , we analyze the case when  $g = 4$  and  $f = 0$  using results on endomorphism rings from Section 4. We thus determine the mod- $\ell$  monodromy group of components of  $\mathcal{H}_g^0$  when  $g \geq 4$ , for all but finitely many  $\ell$ . Note that in Theorem 5.7, the set of exceptional primes depends on the characteristic  $p$  of the base field, but not on  $g$ , so that our results are valid for  $\ell \gg_p 0$ .

**Lemma 5.5.** *Let  $S$  be an irreducible component of  $\mathcal{H}_3^0$ .*

- (a) *Either  $M_{\mathbb{Q}_\ell}(S) \cong \mathrm{Sp}_{2g, \mathbb{Q}_\ell}$  for all  $\ell \neq p$ , or there exists a totally real field  $L$  such that  $M_{\mathbb{Q}_\ell}(S) \cong (\mathbf{R}_{L/\mathbb{Q}} \mathrm{SL}_2) \times \mathbb{Q}_\ell$  for all  $\ell \neq p$ .*
- (b) *Let  $s \in S$  be a geometric point. For  $\ell$  in a set of positive density, there exists a torus  $T_\ell \subset M_{\mathbb{Q}_\ell}(S, s)$  which acts irreducibly on  $V_\ell(\mathcal{X}_s)$ .*

**Proof.** Let  $\eta$  be the generic point of  $S$ , and consider the dichotomy of Lemma 4.1. If  $\mathrm{End}(\mathcal{X}_{3, \eta}) \cong \mathbb{Z}$ , then one knows (e.g., [35, Thm. 3]) that  $M_{\mathbb{Q}_\ell}(S) \cong \mathrm{Sp}_{2g, \mathbb{Q}_\ell}$  for all  $\ell \neq p$ .

Otherwise, if  $\mathbf{E}(\mathcal{X}_{3, \eta}) \cong L$ , a totally real cubic field, then  $S$  coincides with (a component of) the  $p$ -rank zero locus of a Hilbert modular threefold attached to  $L$ . Therefore,  $M_{\mathbb{Q}_\ell}(S) \cong \mathbf{R}_{L/\mathbb{Q}} \mathrm{SL}_2 \times \mathbb{Q}_\ell$  for all  $\ell \neq p$  [39, Lemma 6.5]. This proves (a).

For (b), if  $M_{\mathbb{Q}_\ell}(S)$  is the symplectic group, for each prime  $\ell$  choose a CM field  $K(\ell)$  of degree 6 which is inert at  $\ell$ .

Otherwise, for each rational prime  $\ell$  inert in  $L$ , let  $K(\ell)$  be a totally imaginary quadratic extension of  $L$  which is inert at  $\ell$ .

Then the norm one torus  $T_\ell := (\mathbf{R}_{K(\ell)/\mathbb{Q}} \mathbb{G}_m)^{(1)} \times \mathbb{Q}_\ell$  is a suitable torus.  $\square$

**Proposition 5.6.** *Let  $S$  be an irreducible component of  $\mathcal{H}_4^0$ . For each  $\ell \neq p$ ,  $M_{\mathbb{Q}_\ell}(S) \cong \mathrm{Sp}_{8, \mathbb{Q}_\ell}$ .*

**Proof.** Fix a geometric point  $s \in S$ . By [24, Thm. 3.3], there exists a connected reductive group  $G/\mathbb{Q}$  and an 8-dimensional representation  $V$  of  $G$  such that for  $\ell \gg 0$ , the representation  $M_{\mathbb{Q}_\ell}(S, s) \rightarrow \mathrm{Aut}(V_\ell(\mathcal{X}_{4, s}))$  is isomorphic to the representation  $G \times_{\mathbb{Q}} \mathbb{Q}_\ell \rightarrow \mathrm{Aut}(V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$ . From Zarhin's theorem and the classification of semisimple Lie algebras (see, e.g., [28, Lemma 1.3]), either  $G = \mathrm{Sp}_8$ , or the representation is of Mumford type [25]; and in each case,  $M_{\mathbb{Q}_\ell}(S)$  is in fact isomorphic to  $G \times_{\mathbb{Q}} \mathbb{Q}_\ell$  for all  $\ell \neq p$ . If the representation is of Mumford type, then  $G$  is isogenous to a twist of  $\mathbf{R}_{K/\mathbb{Q}} \mathrm{SL}_2$  for some totally real cubic field  $K$ , and in particular has dimension nine. Therefore, to prove the claim, it suffices to show that  $\dim_{\mathbb{Q}_\ell} M_{\mathbb{Q}_\ell}(S, s) \geq 10$ .

By Corollary 3.14,  $\bar{S}$  intersects  $\Delta_{1,1}(\bar{\mathcal{H}}_{1,1})$ . As in the proof of Theorem 5.2, one can compute  $M_{\mathbb{Q}_\ell}(S, s)$  at a point  $s$  in  $\Delta_{1,1}(\bar{S})$ . Then there are components  $\tilde{S}_L$  and  $\tilde{S}_R$  of  $\tilde{\mathcal{H}}_3^0$ , and components  $\tilde{S}_1$  and  $\tilde{S}_3$  of  $\tilde{\mathcal{H}}_1^0$ , such that  $M_{\mathbb{Q}_\ell}(S, s)$  contains distinct subgroups isomorphic to  $M_{\mathbb{Q}_\ell}(\tilde{S}_L) \times M_{\mathbb{Q}_\ell}(\tilde{S}_3)$  and  $M_{\mathbb{Q}_\ell}(\tilde{S}_1) \times M_{\mathbb{Q}_\ell}(\tilde{S}_R)$ . Moreover, by Lemma 5.5(a), each of  $M_{\mathbb{Q}_\ell}(\tilde{S}_L)$  and  $M_{\mathbb{Q}_\ell}(\tilde{S}_R)$

has dimension at least nine. Therefore,  $\dim_{\mathbb{Q}_\ell} M_{\mathbb{Q}_\ell}(S, s) \geq 10$ , and  $M_{\mathbb{Q}_\ell}(S, s) \cong \mathrm{Sp}_{8, \mathbb{Q}_\ell}$  for all  $\ell \neq p$ .  $\square$

**Theorem 5.7.** *If  $\ell \gg_p 0$ , if  $g \geq 4$  and if  $S$  is an irreducible component of  $\mathcal{H}_g^0$ , then  $M_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$  and  $M_{\mathbb{Z}_\ell}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ .*

**Proof.** The proof is by induction on  $g$ . For the base case, suppose  $g = 4$ . Since  $\mathcal{H}_4^0$  has only finitely many irreducible components, it suffices to prove the statement for  $\ell \gg_p 0$  for each component  $S$ . This follows from Proposition 5.6 and, e.g., [35, 8.2].

For  $g > 4$ , the proof is identical to that of Theorem 5.2, with Corollary 3.14 being used to degenerate to a component of  $\tilde{\kappa}_{1, g-2, 1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_{g-2}^0 \times \tilde{\mathcal{H}}_1^0)$ .  $\square$

**Remark 5.8.** The assertion of Theorem 5.7 is false for  $\mathcal{H}_1^0$  and  $\mathcal{H}_2^0$  if  $\ell \geq 5$ . Indeed, a hyperelliptic curve of genus  $g \leq 2$  and  $p$ -rank 0 is supersingular. Since a supersingular  $p$ -divisible group over a scheme  $S$  becomes trivial after a finite pullback  $\tilde{S} \rightarrow S$ , the monodromy group  $M_{\mathbb{Z}_\ell}(\mathcal{H}_g^0)$  is finite for  $g \leq 2$ . The ambiguity in Lemma 4.1 propagates to Lemma 5.5, and we do not know whether the assertion of Theorem 5.7 is true for  $\mathcal{H}_3^0$ .

## 5.4. Arithmetic applications

The results of the previous section about the monodromy of components of  $\mathcal{H}_g^f$  have arithmetic applications involving hyperelliptic curves over finite fields. For example, they imply that there exist hyperelliptic curves of given genus and  $p$ -rank with absolutely simple Jacobian (Application 5.9). Moreover, they give estimates for the proportion of hyperelliptic curves with a given genus and  $p$ -rank which have a rational point of order  $\ell$  on the Jacobian (Application 5.11) or for which the numerator of the zeta function has large splitting field (Application 5.13).

Throughout this section,  $\mathbb{F}$  denotes a finite extension of  $\mathbb{F}_p$ .

### 5.4.1. Technical context

We do not include proofs in this section, since they are very similar to those found in [4, Section 5]. Here is a brief description of the main ideas involved. One first defines  $\mathcal{H}_g$  over the category of  $\mathbb{F}_p$ -schemes and defines the arithmetic monodromy group of a substack of  $\mathcal{H}_g$ . For a relative curve  $\pi : C \rightarrow S/\mathbb{F}$  of genus  $g \geq 2$  defined over a finite field, one shows that if  $M_\ell^{\mathrm{geom}}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ , then  $M_{\mathbb{Z}_\ell}^{\mathrm{geom}}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ ; and  $M_{\mathbb{Z}_\ell}^{\mathrm{arith}}(S)$  has finite index in  $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$ ; and  $M_{\mathbb{Q}_\ell}^{\mathrm{geom}}(S) \cong \mathrm{Sp}_{2g, \mathbb{Q}_\ell}$  [4, Lemma 5.1].

Secondly, in order to use Chebotarev arguments for curves over finite fields, it is necessary to add rigidifying data, such as the data of a tricanonical structure, so that the corresponding moduli problems are representable by schemes. Recall that  $\Omega_{C/S}^{\otimes 3}$  is very ample, that  $\pi_*(\Omega_{C/S}^{\otimes 3})$  is a locally free  $\mathcal{O}_S$ -module of rank  $5g - 5$ , and that sections of this bundle define a closed embedding  $C \hookrightarrow \mathbb{P}_S^{5g-5}$ . A tricanonical (3K) structure on  $\pi : C \rightarrow S$  is a choice of isomorphism  $\mathcal{O}_S^{\oplus 5g-5} \cong \pi_*(\Omega_{C/S}^{\otimes 3})$ , and the only automorphisms of a hyperelliptic curve with 3K-structure are the identity and the hyperelliptic involution. The moduli space  $\mathcal{H}_{g, 3K}$  of smooth hyperelliptic curves of genus  $g$  equipped with a 3K-structure is representable by a scheme [21, 10.6.5], [26, Prop. 5.1].

Third, since  $\mathcal{H}_g$  may be constructed as the quotient of  $\mathcal{H}_{g, 3K}$  by  $\mathrm{GL}_{5g-5}$ , the forgetful functor  $\psi_g : \mathcal{H}_{g, 3K} \rightarrow \mathcal{H}_g$  is open [26, p. 6] and a fibration with connected fibers [27, Thm. A.12].



Thus, if  $S \subset \mathcal{H}_g$  is a connected substack and  $S_{3K} = S \times_{\mathcal{H}_g} \mathcal{H}_{g,3K}$ , then  $M_\ell(S_{3K}) \cong M_\ell(S)$  [4, Lemma 5.2]. Since the data of a tricanonical structure exists Zariski-locally on the base, one can relate point counts on  $\mathcal{H}_{g,3K}(\mathbb{F})$  to those on  $\mathcal{H}_g(\mathbb{F})$ . Specifically, if  $s \in \mathcal{H}_g^f(\mathbb{F})$  is such that  $\text{Aut}(\mathcal{C}_{g,s}) \cong \{\pm 1\}$ , then the fiber of  $\mathcal{H}_{g,3K}(\mathbb{F})$  over  $s$  consists of  $|\text{GL}_{2g}(\mathbb{F})|/2$  points [21, 10.6.8].

#### 5.4.2. Application to simple Jacobians

Using the  $\mathbb{Q}_\ell$ -monodromy of  $\mathcal{H}_g^f$ , we deduce that there exist hyperelliptic curves of genus  $g$  and  $p$ -rank  $f$  with absolutely simple Jacobian.

**Application 5.9.** Suppose  $g \geq 1$  and  $0 \leq f \leq g$  with  $f \neq 0$  if  $g \leq 2$ . Let  $S$  be an irreducible component of  $\mathcal{H}_g^f$ . Then there exists  $s \in S(\overline{\mathbb{F}})$  such that the Jacobian of  $\mathcal{C}_{g,s}$  is absolutely simple.

**Proof.** If  $f \geq 1$  or  $g \geq 4$ , then this follows from Theorems 5.2 and 5.7 using [9, Prop. 4]. If  $f = 0$  and  $g = 3$ , then this follows from Lemma 5.5 using [9, Rem. 5(i)].  $\square$

**Remark 5.10.** Note that, for arbitrary  $g$ , it is unknown how to deduce Application 5.9 from Theorem 4.6. Also, under the hypotheses of Application 5.9, one can deduce that  $\text{Aut}(\mathcal{C}_{g,\eta}) = \{\pm 1\}$ , which yields a new proof of [2, Thm. 3.7].

#### 5.4.3. Application to class groups

Recall that if  $s \in \mathcal{H}_g(\mathbb{F})$ , then  $\text{Pic}^0(\mathcal{C}_{g,s})(\mathbb{F})$  is isomorphic to the class group of the function field  $\mathbb{F}(\mathcal{C}_{g,s})$ . The size of the class group is divisible by  $\ell$  exactly when there is a point of order  $\ell$  on the Jacobian. Roughly speaking, Application 5.11 shows that among all curves over  $\mathbb{F}$  of specified genus and  $p$ -rank, slightly more than  $1/\ell$  of them have an  $\mathbb{F}$ -rational point of order  $\ell$  on their Jacobian.

**Application 5.11.** Suppose  $\ell$  and  $p$  are distinct odd primes,  $g \geq 1$  and  $1 \leq f \leq g$ . Suppose  $S$  is an irreducible component of  $\mathcal{H}_g^f$  such that  $S(\mathbb{F}) \neq \emptyset$ . Let  $m$  be the image of  $|\mathbb{F}|$  in  $(\mathbb{Z}/\ell)^\times$ . There exist a rational function  $\alpha_{g,m}(T) \in \mathbb{Q}(T)$  and a constant  $B = B(p, g, \ell)$  such that

$$\left| \frac{\#\{s \in S(\mathbb{F}) : \ell \mid \#\text{Pic}^0(\mathcal{C}_{g,s})(\mathbb{F})\}}{\#S(\mathbb{F})} - \alpha_{g,m}(\ell) \right| < \frac{B}{\sqrt{|\mathbb{F}|}}. \quad (5.4.1)$$

If  $f = 0$  and  $g \geq 4$ , the same result is true for all  $\ell \gg_p 0$ .

**Proof.** The proof is very similar to that of [4, Application 5.9] and uses Theorems 5.2–5.7 and [21, Thm. 9.7.13].  $\square$

**Remark 5.12.** For  $\ell$  odd, one knows that  $\alpha_{g,1}(\ell) = \frac{\ell}{\ell^2-1} + \mathcal{O}(1/\ell^3)$ , while  $\alpha_{g,m}(\ell) = \frac{1}{\ell-1} + \mathcal{O}(1/\ell^3)$  if  $m \neq 1$ . A formula for  $\alpha_{g,1}(\ell)$  is given in [1].

Note that Application 5.11 shows that for a given  $g$ , the distribution of (arithmetic)  $\ell$ -ranks among hyperelliptic curves over  $\mathbb{F}$  with fixed (geometric)  $p$ -rank  $f$  is essentially independent of  $f$ . The reader interested in the field of definition of  $\ell$ -torsion in the Jacobians of individual hyperelliptic curves is invited to consult [5] and the discussion therein. To the best of our knowledge, none of the works cited there directly analyzes the (lack of) interplay between  $\ell$ - and  $p$ -ranks.

#### 5.4.4. Application to zeta functions

If  $C/\mathbb{F}$  is a smooth projective curve of genus  $g$ , its zeta function has the form  $L_{C/\mathbb{F}}(T)/(1-T)(1-|\mathbb{F}|T)$ , where  $L_{C/\mathbb{F}}(T) \in \mathbb{Z}[T]$  is a polynomial of degree  $2g$ . The principal polarization on the Jacobian of  $C$  forces a symmetry among the roots of  $L_{C/\mathbb{F}}(T)$ ; the largest possible Galois group for the splitting field over  $\mathbb{Q}$  of  $L_{C/\mathbb{F}}(T)$  is the Weyl group of  $\mathrm{Sp}_{2g}$  which is a group of size  $g!2^g$ .

**Application 5.13.** Suppose  $g \geq 1$  and  $1 \leq f \leq g$ , or that  $g \geq 4$  and  $f = 0$ . Suppose  $p > 2g + 1$  and that  $S$  is an irreducible component of  $\mathcal{H}_g^f$  such that  $S(\mathbb{F}) \neq \emptyset$ . There exist a constant  $\gamma = \gamma(g) > 0$  and a constant  $E = E(p, g)$  such that

$$\frac{\#\{s \in S(\mathbb{F}) : L_{C_{g,s}/\mathbb{F}}(T) \text{ is reducible, or has splitting field with degree } < 2^g g!\}}{\#S(\mathbb{F})} < E\mathbb{F}^{-\gamma}. \quad (5.4.2)$$

**Proof.** The proof is very similar to that of [4, Application 5.1.1] and uses Theorems 5.2–5.7 and [23, Thm. 6.1 and Remark 3.2.(4)].  $\square$

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